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Universal few-body physics

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Aim

Look at a particular aspect of quantum physics: the **universal physics** that arises for **nearly-resonant short-range interactions**.

> nuclear physics

> atomic physics (cold atoms)

> condensed matter, etc.

Look in detail at the case of two-body and three-body systems, and in particular the **Efimov effect**.

Plan

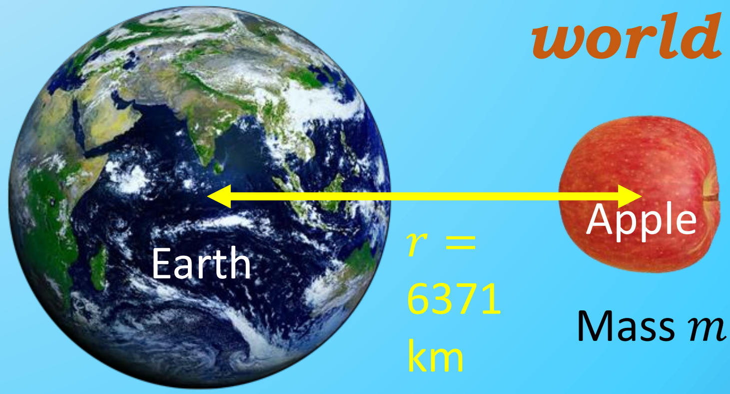
1. **INTRODUCTION**: WHAT IS UNIVERSALITY?
2. **TWO-BODY PHYSICS**: RESONANCES AND ZERO-RANGE UNIVERSALITY
3. **THREE-BODY PHYSICS**: A HISTORY OF THE THREE-BODY PROBLEM AND THE EFIMOV EFFECT
4. **ZERO-RANGE UNIVERSALITY**: AN OVERVIEW OF UNIVERSAL CLUSTERS
5. **VAN DER WAALS UNIVERSALITY**

What is universality?

A phenomenon is ***universal*** when it applies to *many different physical systems* with a simple law that depends upon just *a few parameters*.

Gravitation is universal

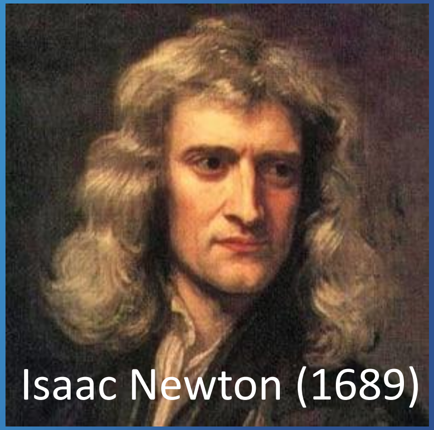
Terrestrial world



$r =$ 6371 km

Mass $m = 0.1$ kg

Mass $M = 6 \times 10^{24}$ kg



Isaac Newton (1689)

Universal law: long-range force

$$V(r) = -G\frac{Mm}{r}$$

Celestial world

$R =$ 384,400 km

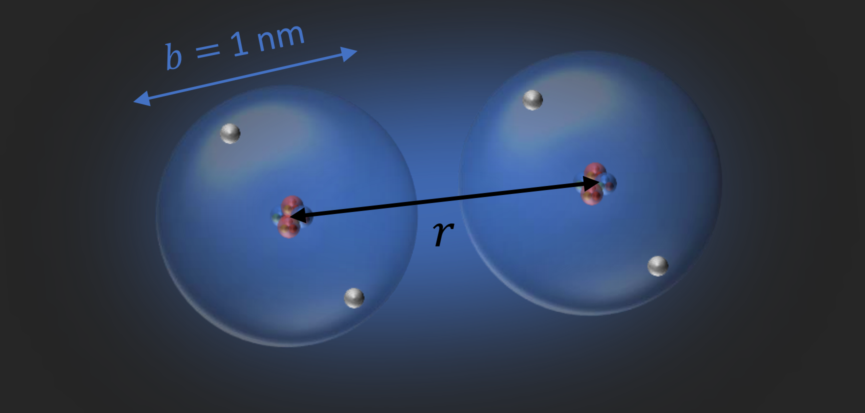
Mass $M' = 7 \times 10^{22}$ kg

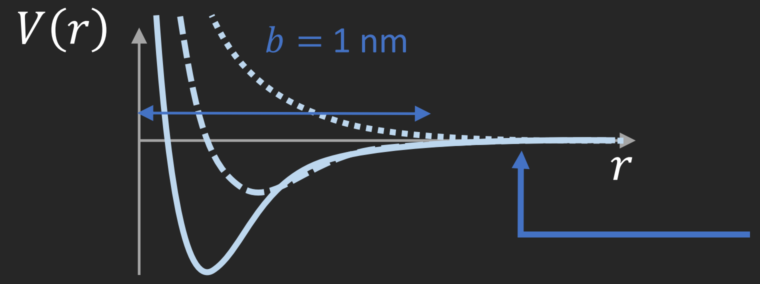
$$V(R) = -G\frac{MM'}{R}$$

No apparent universality in the microscopic world

- Short-range forces
- Many different interaction potentials that depend on the nature and states of the particles

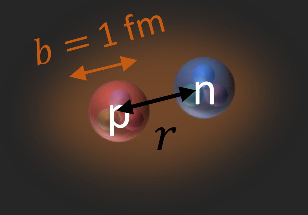
Atomic world Electromagnetic force

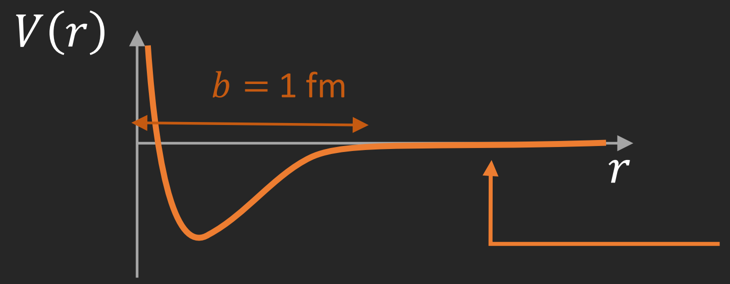




$$-\frac{C_6}{r^6}$$ (van der Waals)

Subatomic world Strong force





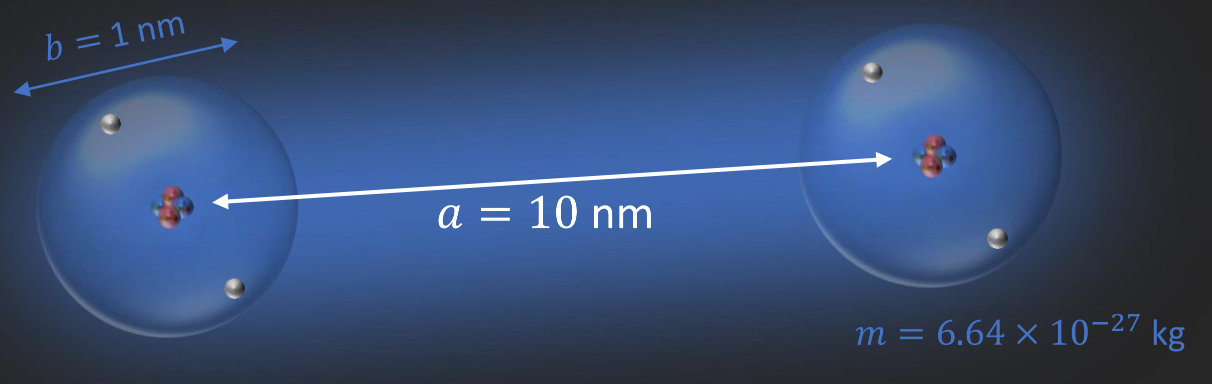
$$-\frac{e^{-\alpha r}}{r}$$ (Yukawa)

“Zero-range” universality

Approximate universal law

Atomic world Electromagnetic force

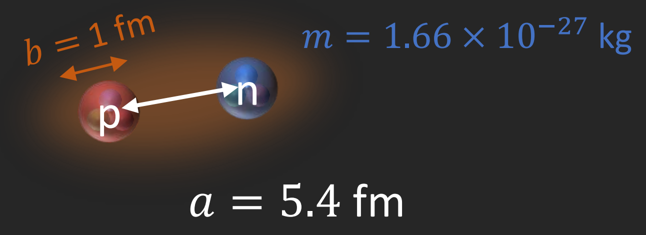
Helium-4 dimer



$$E \approx -\frac{\hbar^2}{ma^2} = 1.0 \times 10^{-7} \text{ eV}$$

Subatomic world Strong force

Deuteron



$$E \approx -\frac{\hbar^2}{ma^2} = 1.4 \times 10^{6} \text{ eV}$$

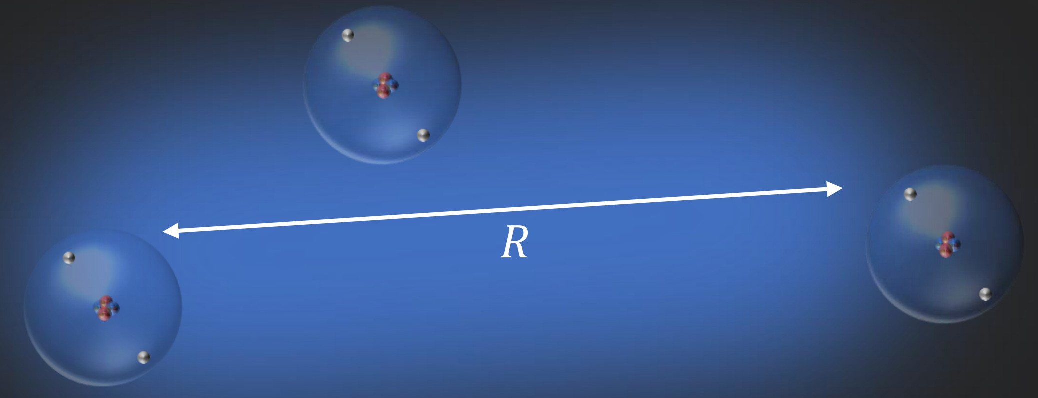
“Zero-range” universality

Approximate universal law: effective long-range force

Efimov attraction

Atomic world Electromagnetic force

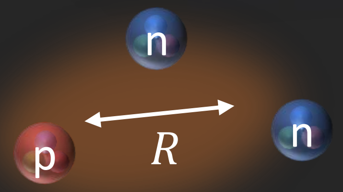
Helium-4 trimer



$$V(R) \approx -\frac{\hbar^2}{mR^2}$$

Subatomic world Strong force

Triton

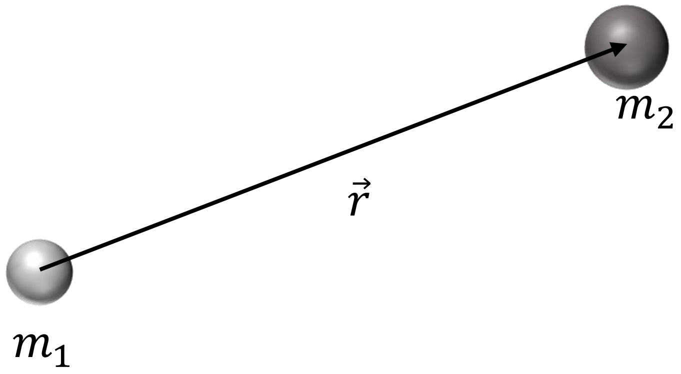


$$V(R) \approx -\frac{\hbar^2}{mR^2}$$

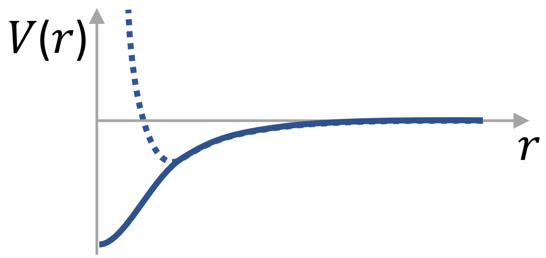
Two-body physics

Basic theory

We consider two particles of masses m_1 and m_2 with no internal degree of freedom.
In their centre of mass, they are described by a relative vector $\vec{r}$.

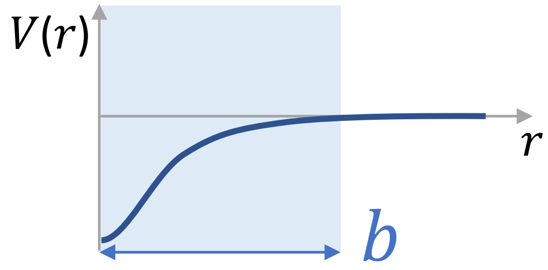


The particles interact via an isotropic (central) interaction potential $V(r)$, with $r = |\vec{r}|$



Attractive

⇐ Necessary for binding



Short range

⇔ $V(r)$ decays strictly faster than r^{-3}

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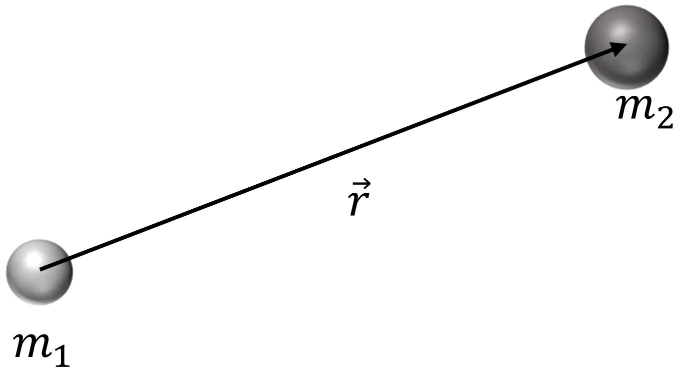

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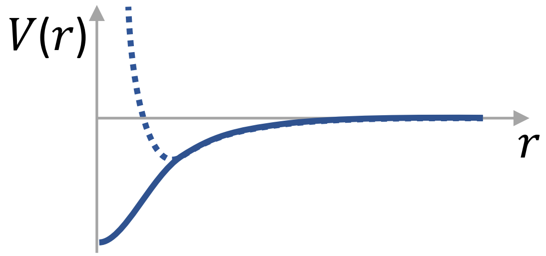

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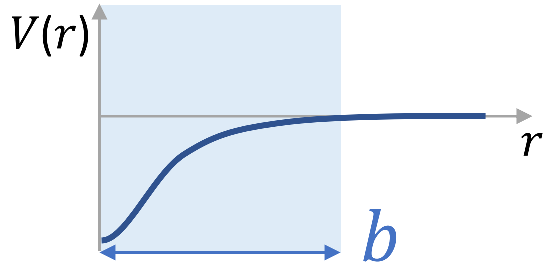

Short range

↔ $V(r)$ decays strictly faster than r^{-3}

Classically:

$$\underbrace{\frac{1}{2}\mu\left(\frac{d\vec{r}}{dt}\right)^2}_{\text{Kinetic energy}} + \underbrace{V(r)}_{\text{Interaction energy}} = \underbrace{E}_{\text{Total energy}}$$

Reduced mass: $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}$

Basic theory

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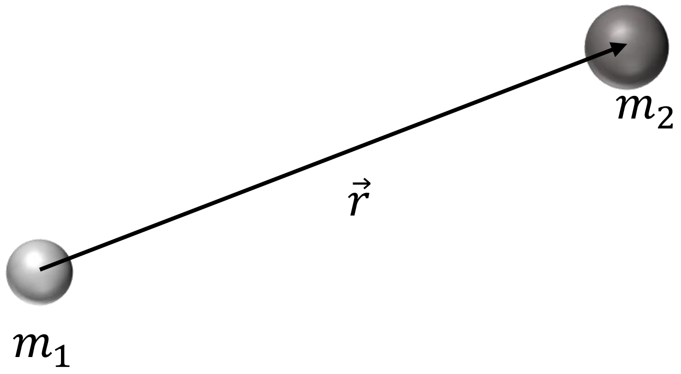

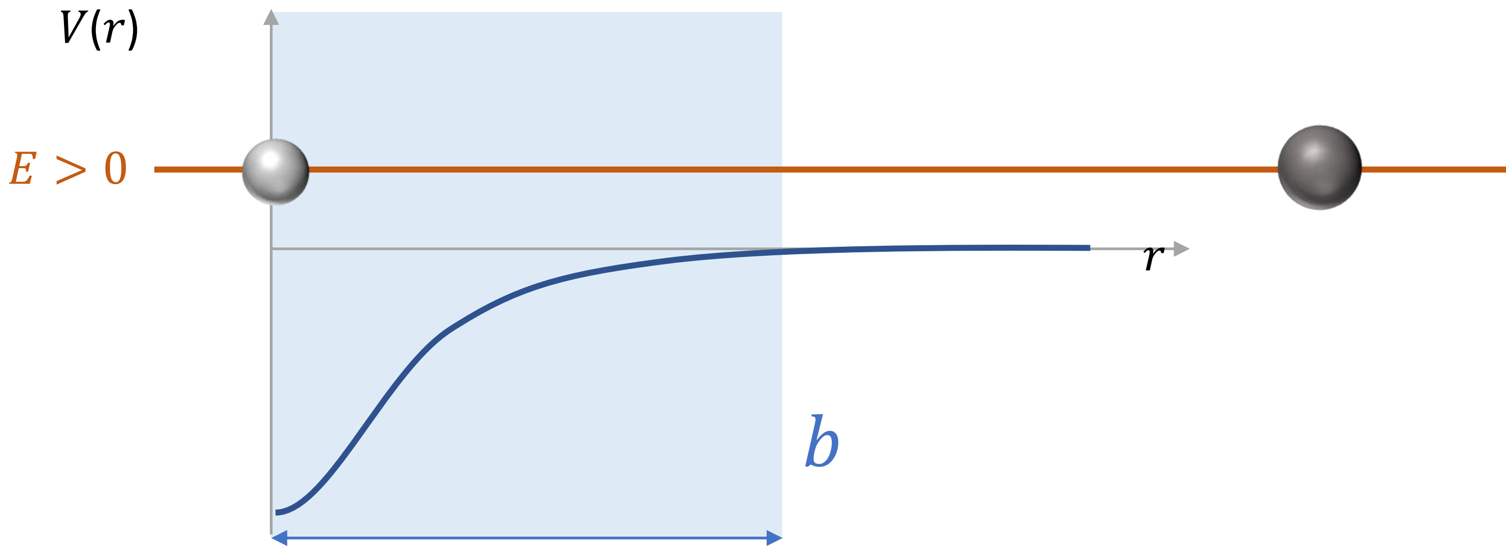

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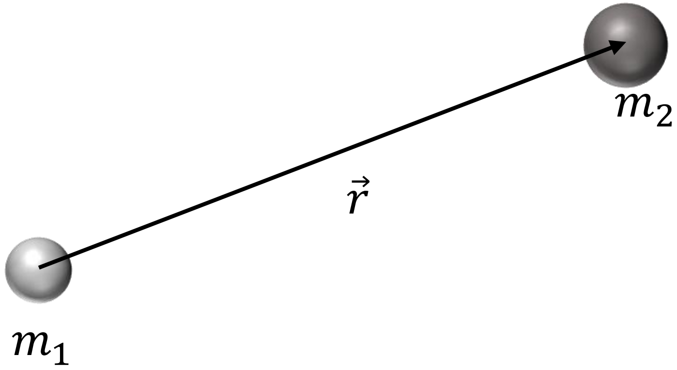

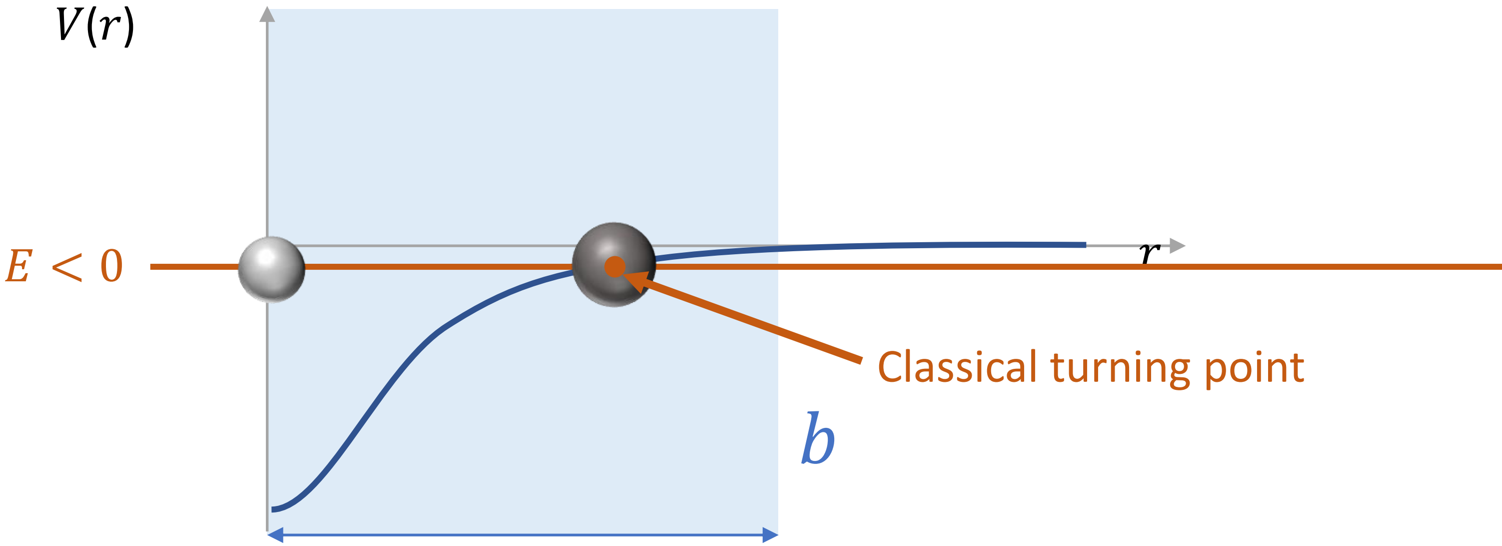

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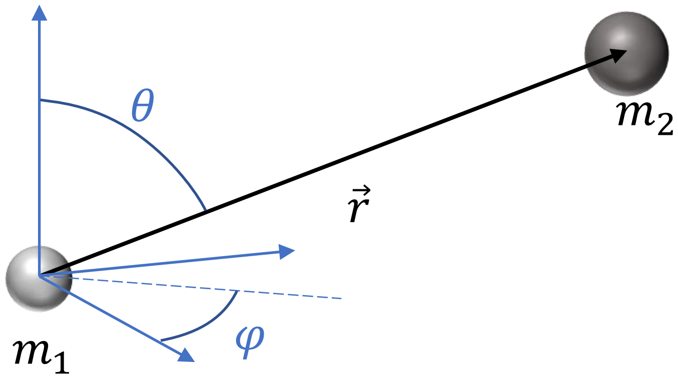


Quantum mechanically, the vector $\vec{r}$ is described by a wave function $\psi(\vec{r})$.

Schrödinger equation at energy E: $$\left(-\frac{\hbar^2}{2\mu}\nabla_{\boldsymbol{r}}^2 + V(r) - E\right)\psi(\vec{r}) = 0$$

$$\left(\underbrace{-\nabla_{\boldsymbol{r}}^2}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\cdot) + \frac{1}{r^2}\left(\underbrace{\cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\theta^2}}_{\ell(\ell+1)} + \cancel{\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}}\right)\underbrace{P_\ell(\cos\theta)}_{\text{Legendre polynomial}}} + \frac{2\mu}{\hbar^2}V(r) - \frac{2\mu}{\hbar^2}E\right)\psi(\vec{r}) = 0$$

Spherical coordinates: $\psi(\vec{r}) = \psi(r, \theta, \cancel{\varphi})$

Partial wave expansion: $= \sum_{\ell=0}^{\infty} \frac{u_\ell(r)}{r} P_\ell(\cos\theta)$

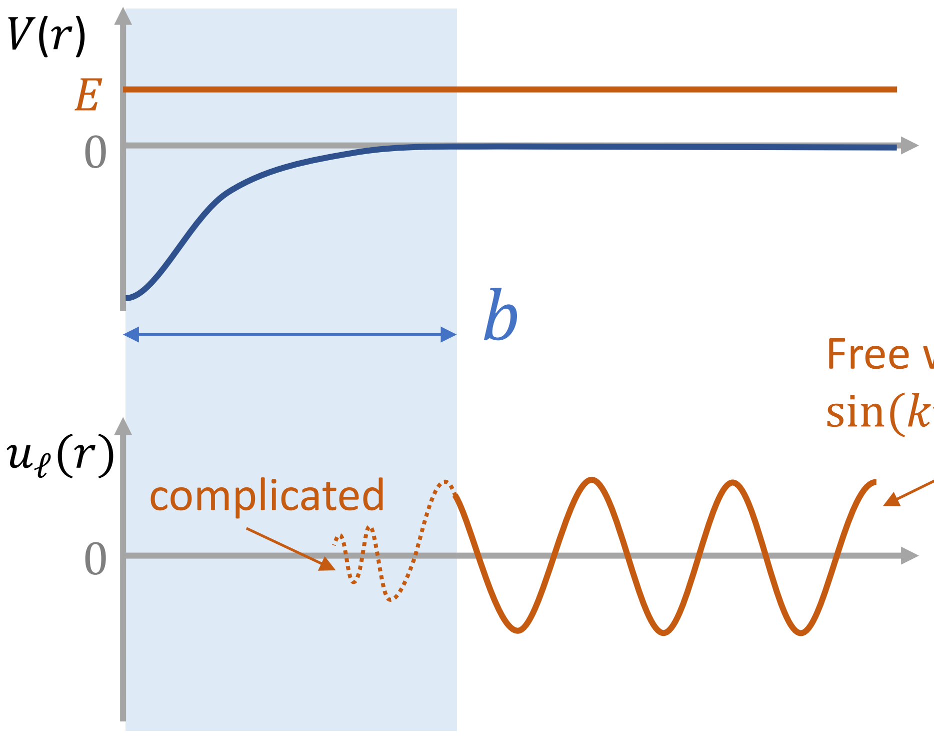
$\ell = 0$: s wave
$\ell = 1$: p wave
$\ell = 2$: d wave

Radial Schrödinger equations: $$\left(\underbrace{-\frac{d^2}{dr^2}}_{\text{Radial kinetic energy}} + \underbrace{\frac{\ell(\ell+1)}{r^2}}_{\text{Centrifugal repulsion}} + \frac{2\mu}{\hbar^2}V(r) - \frac{2\mu}{\hbar^2}E\right)u_\ell(r) = 0$$

Basic theory

Radial Schrödinger equations: $\left(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{-k^2}\right)u_\ell(r)=0$ k: wave number

Positive energy $\boldsymbol{E=\frac{\hbar^2k^2}{2\mu}\geq 0}$



Free wave:
$\sin(kr-\ell\pi/2+\delta_\ell(k))$

$\delta_\ell=0$ modulo π : effectively non-interacting
$\delta_\ell\approx\frac{\pi}{2}$ modulo π : **resonant interaction**

At low scattering energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta_0\approx -ka}$

"Scattering length"
(positive or negative)

Low-energy s-wave **resonance**: $|a|\gg b$

Unitarity (or ***unitary limit***): $a\rightarrow\pm\infty$

Cross section: $\sigma_0=\frac{4\pi}{k^2}\sin^2\delta_0\leq\frac{4\pi}{k^2}$

Basic theory

Radial Schrödinger equations:

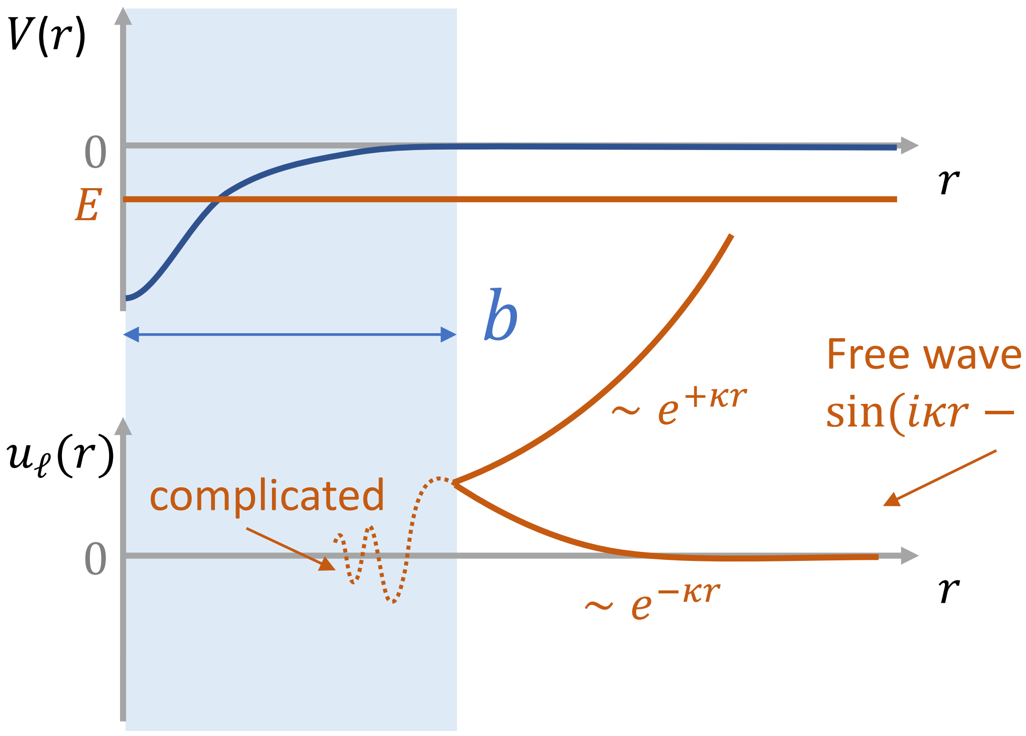
$$\left(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{+\kappa^2}\right)u_\ell(r)=0$$

$$\boxed{k\to i\kappa}$$

$\kappa\geq 0$: binding wave number

Negative energy $\boldsymbol{E=-\frac{\hbar^2\kappa^2}{2\mu}\leq 0}$

$$\sin(kr-\ell\pi/2+\delta_\ell)$$
$$\propto \sin(kr-\ell\pi/2)+\tan\delta_\ell\cos(kr-\ell\pi/2)$$
$$\propto \left(\tan\delta_\ell+\frac{1}{i}\right)e^{i\left(kr-\frac{\ell\pi}{2}\right)}+\left(\tan\delta_\ell-\frac{1}{i}\right)e^{-i\left(\kappa r-\frac{\ell\pi}{2}\right)}$$
$$\propto \left(\tan\delta_\ell+\frac{1}{i}\right)(-i)^\ell e^{-\kappa r}+\underbrace{\left(\tan\delta_\ell-\frac{1}{i}\right)}_{0}(i)^\ell e^{+\kappa r}$$



Free wave:
$\sin(i\kappa r-\ell\pi/2+\delta_\ell(i\kappa))$

Physical state (bound state) : $\tan\delta_\ell=\frac{1}{i}$
(quantisation of the energy: discrete levels)

For small energy E, in the s wave:

For $k\ll b^{-1}$, $\boxed{\tan\delta_0\approx -ka=-i\kappa a}$

➡ $\kappa\approx 1/a$ $\boxed{E\approx-\frac{\hbar^2}{2\mu a^2}}$ (For $|a|\gg b$)

Basic theory

Radial Schrödinger equations: $\left(-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+\frac{2\mu}{\hbar^2}V(r)\underbrace{-\frac{2\mu}{\hbar^2}E}_{+\kappa^2}\right)u_\ell(r)=0$

$\boxed{k \to i\kappa}$

$\kappa \geq 0$: binding wave number

Negative energy $E=-\frac{\hbar^2\kappa^2}{2\mu}\leq 0$

$$\sin(kr-\ell\pi/2+\delta_\ell)$$
$$\propto \sin(kr-\ell\pi/2)+\tan\delta_\ell\cos(kr-\ell\pi/2)$$
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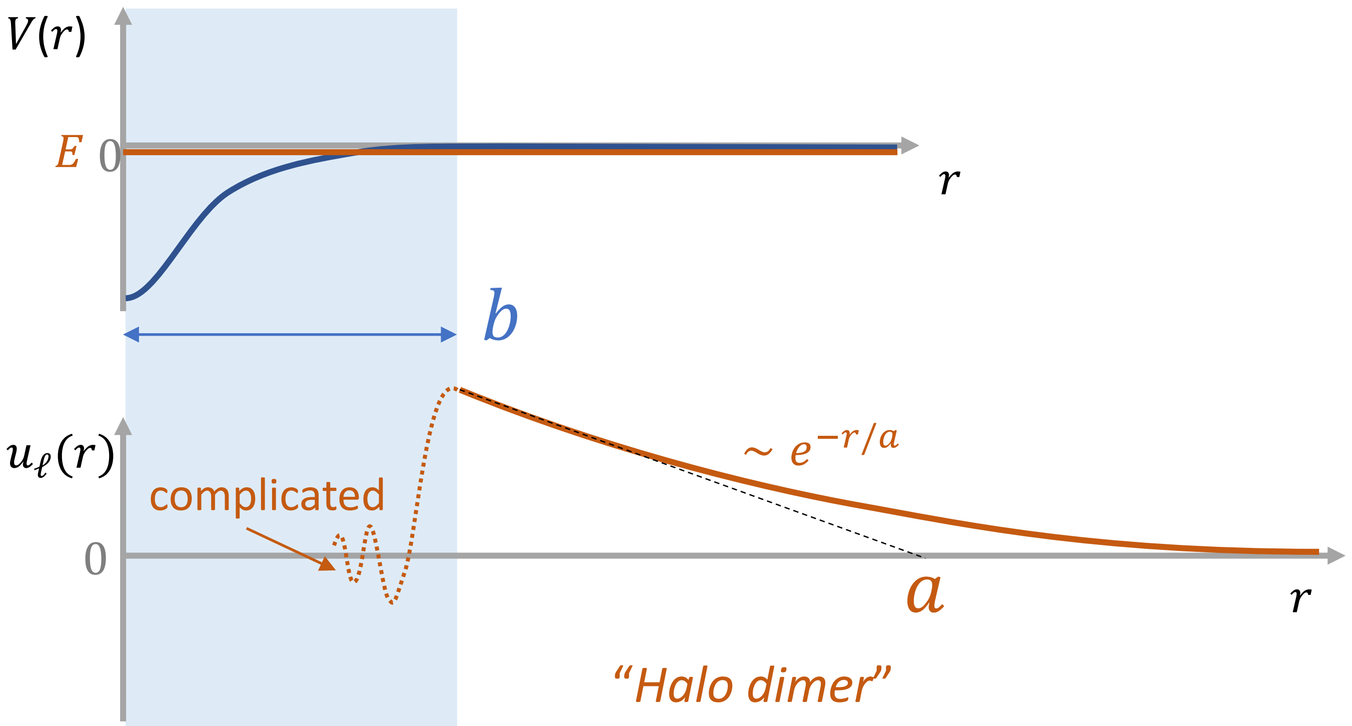

"Halo dimer"

Extends beyond the classical turning point

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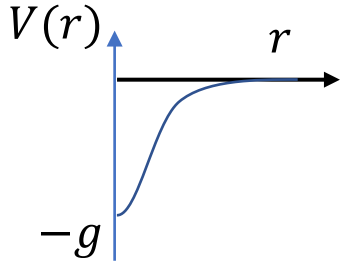
Short-range attractive interaction potential



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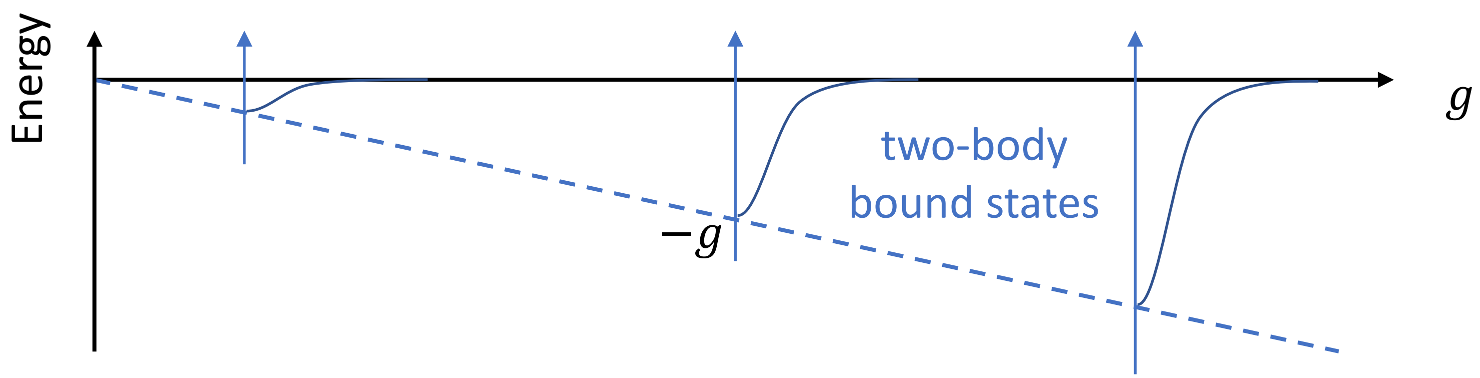
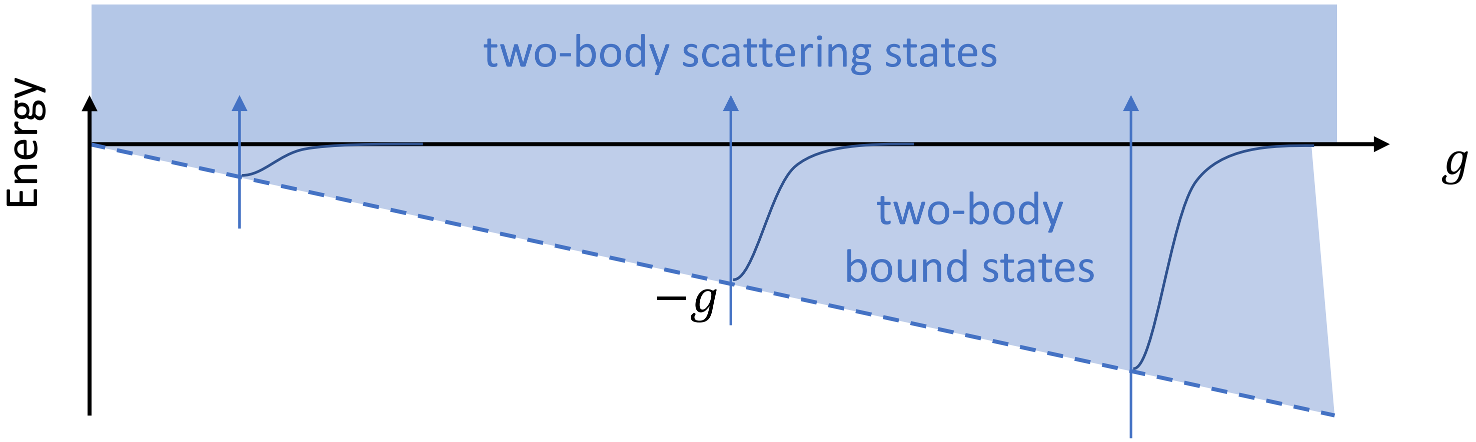
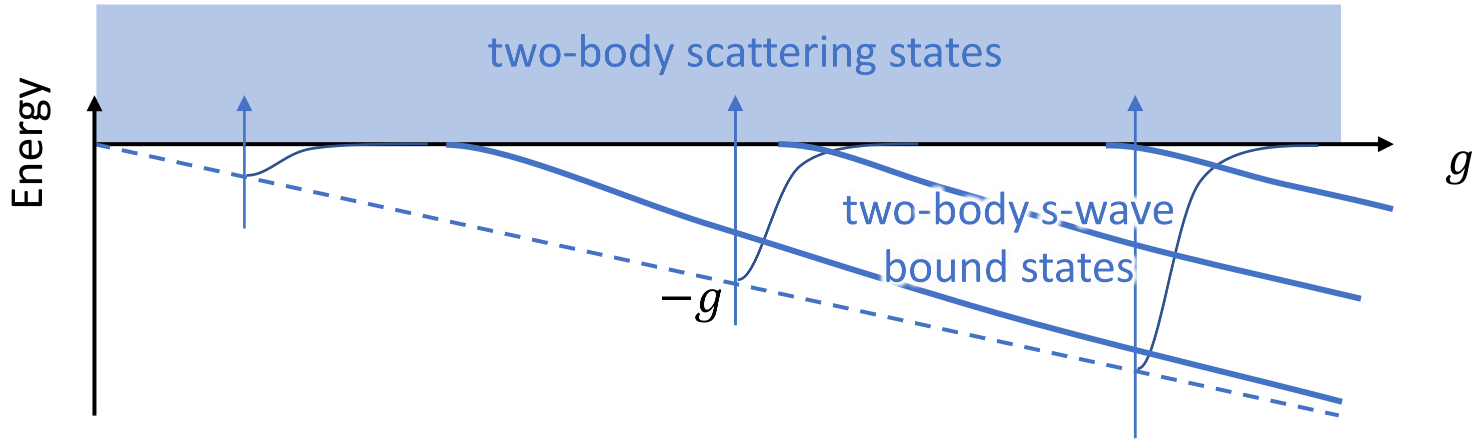


Binding in classical systems



Binding in quantum systems

Critical strength g (zero-point) and quantisation



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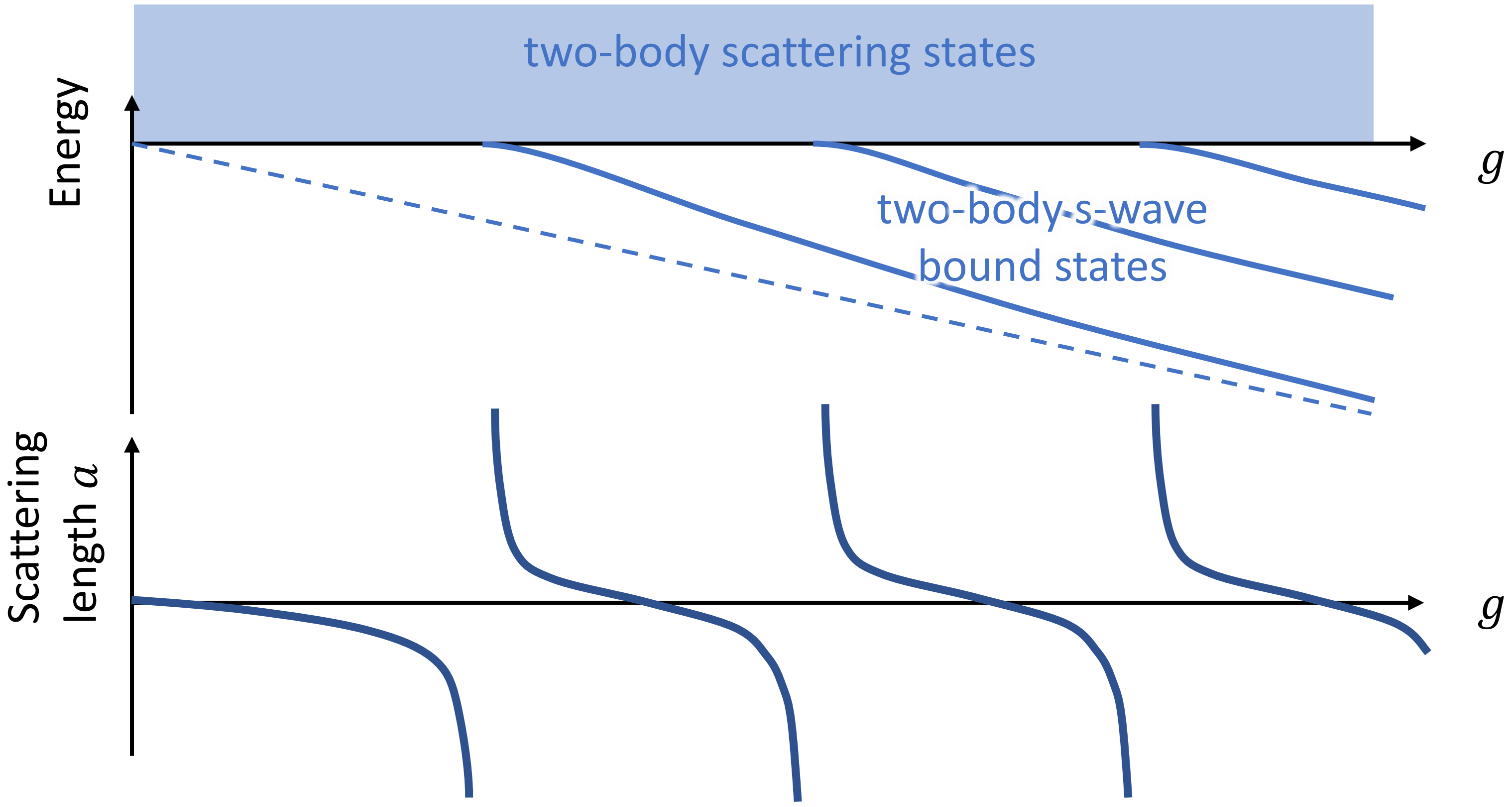
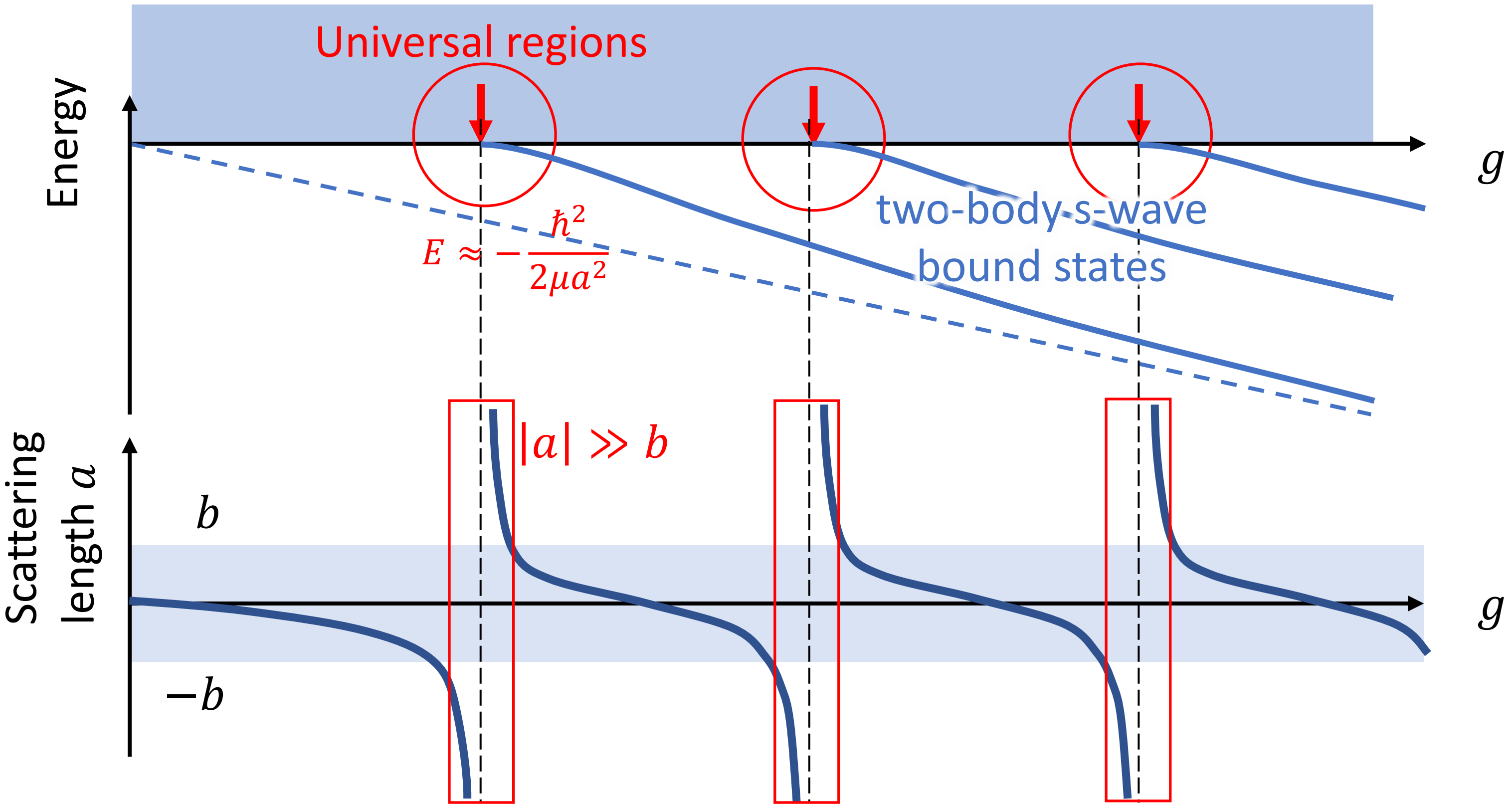


Binding in quantum systems

Two-body resonances $\Longrightarrow$ unitarity points, universality and scale invariance



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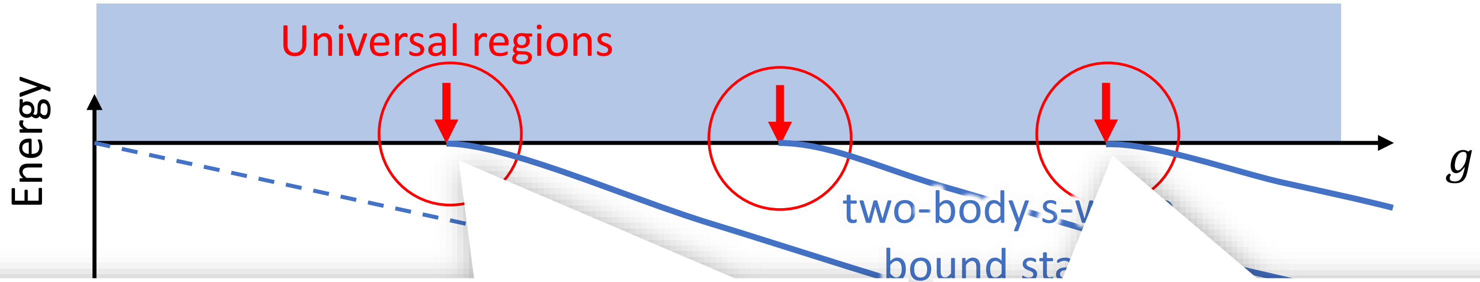
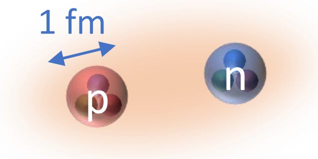


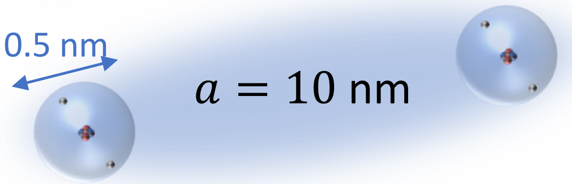
In nature

Deuteron



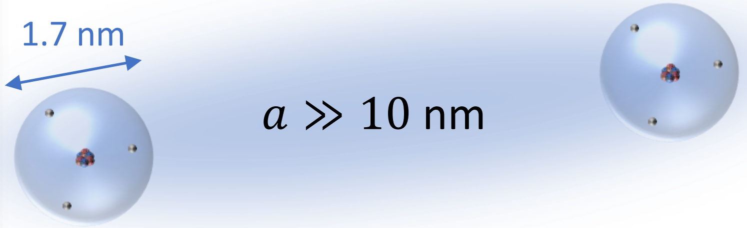
$a = 5.4$ fm

Helium-4 dimer



In the laboratory

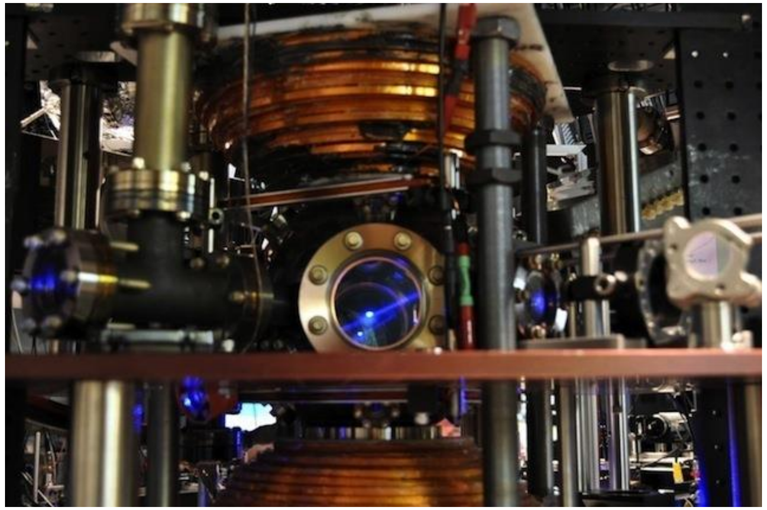
Feshbach dimers from cold atoms

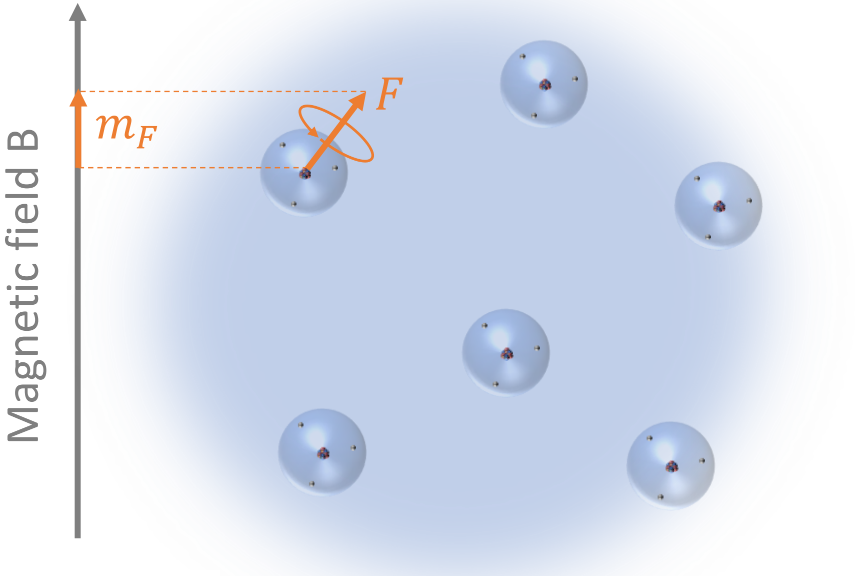


“Halo dimers”

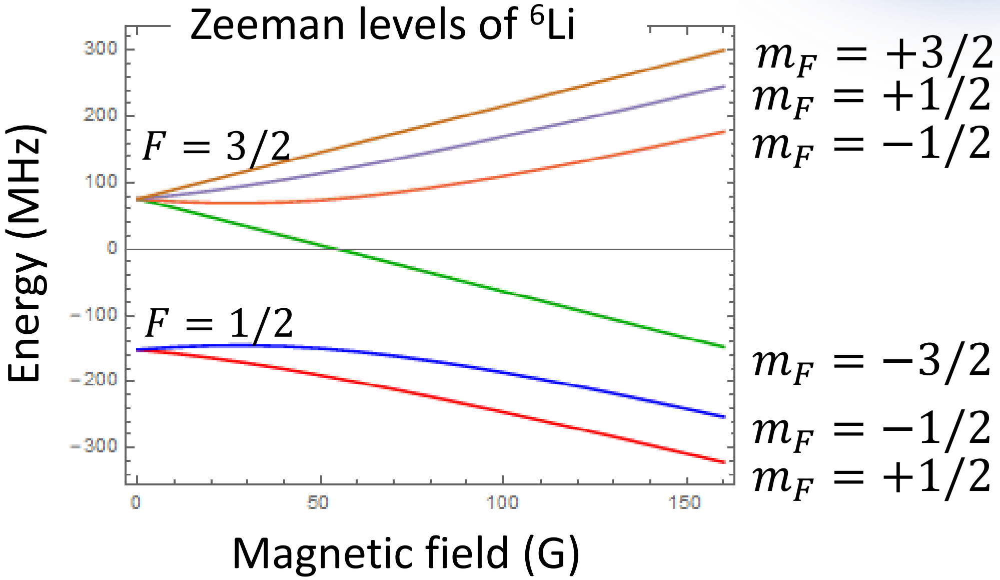
Observations in ultra-cold atomic gases

Cloud of atoms cooled to T < 1 μK in a vacuum chamber





Zeeman effect: Different internal atomic states shift differently with magnetic field



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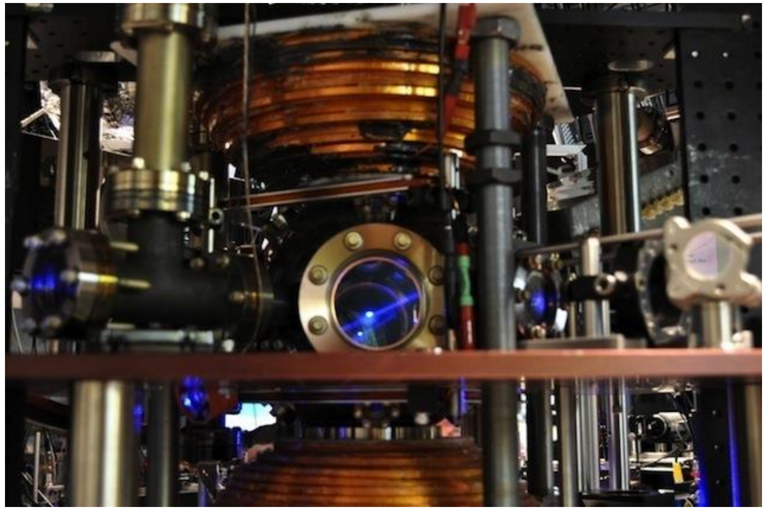

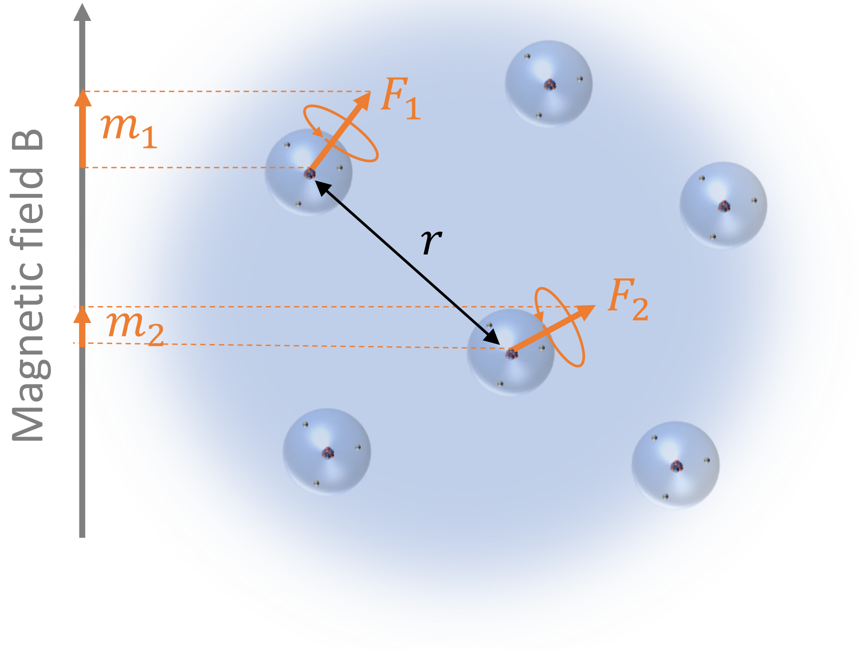

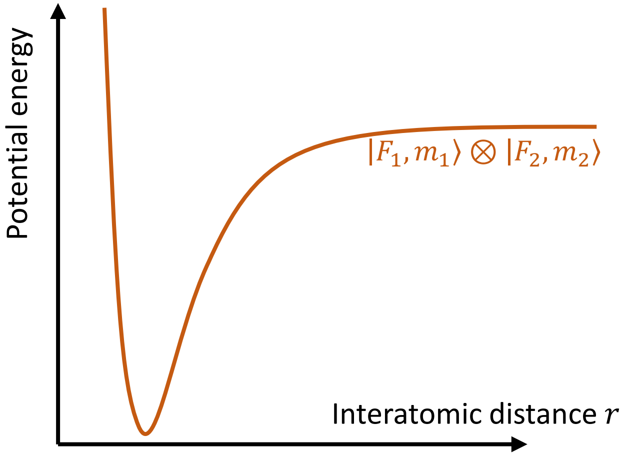

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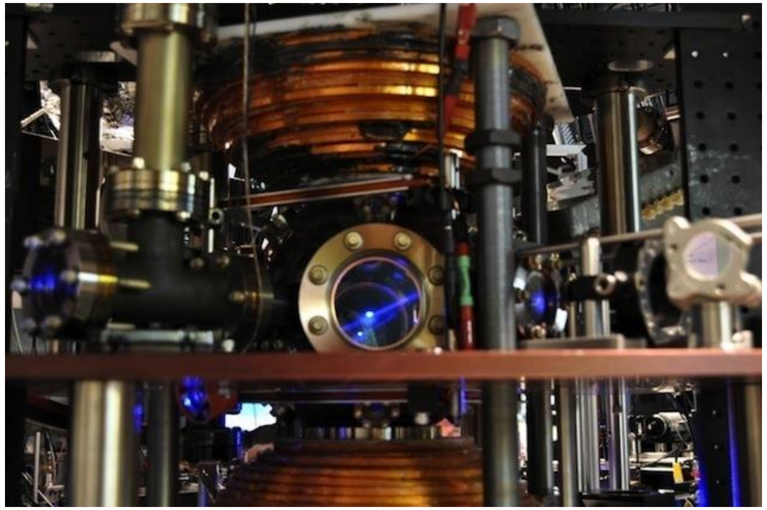

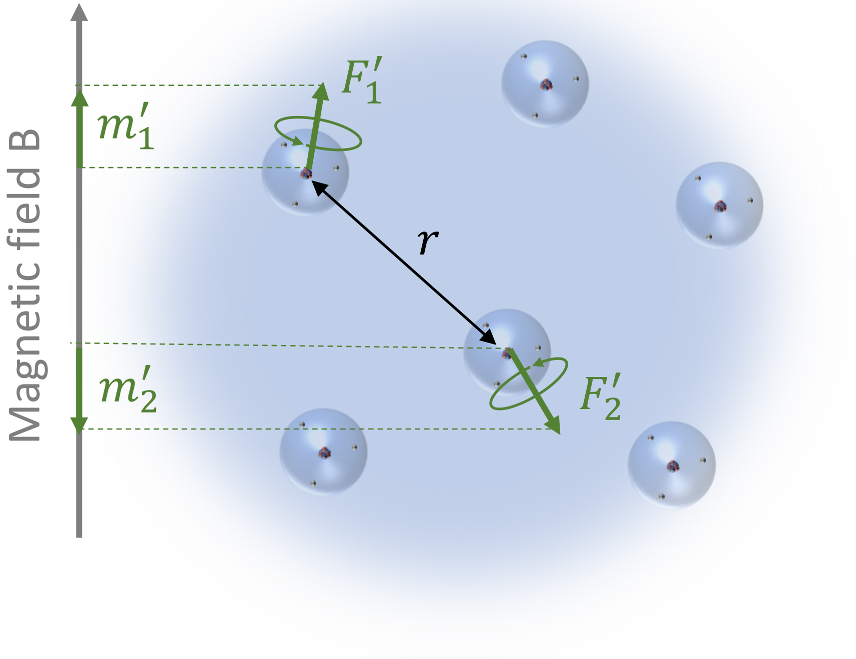

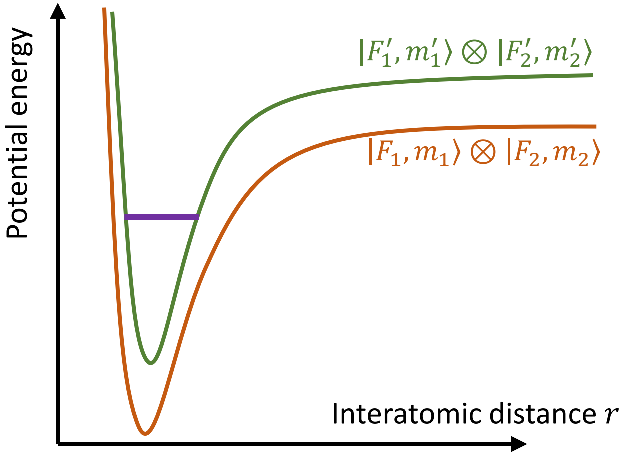

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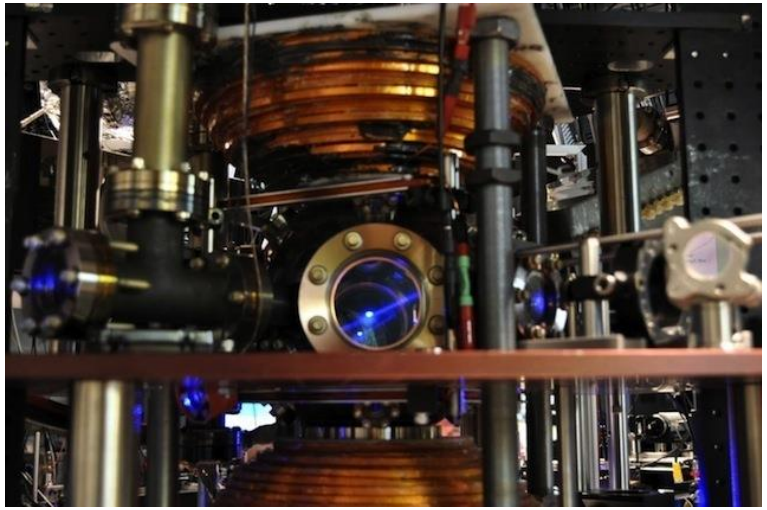

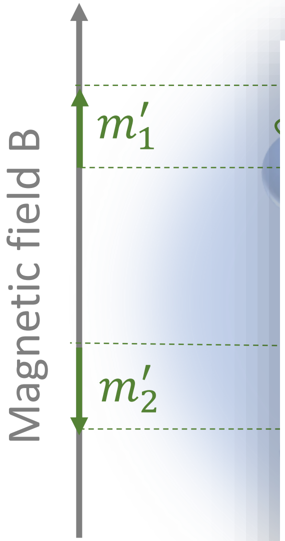

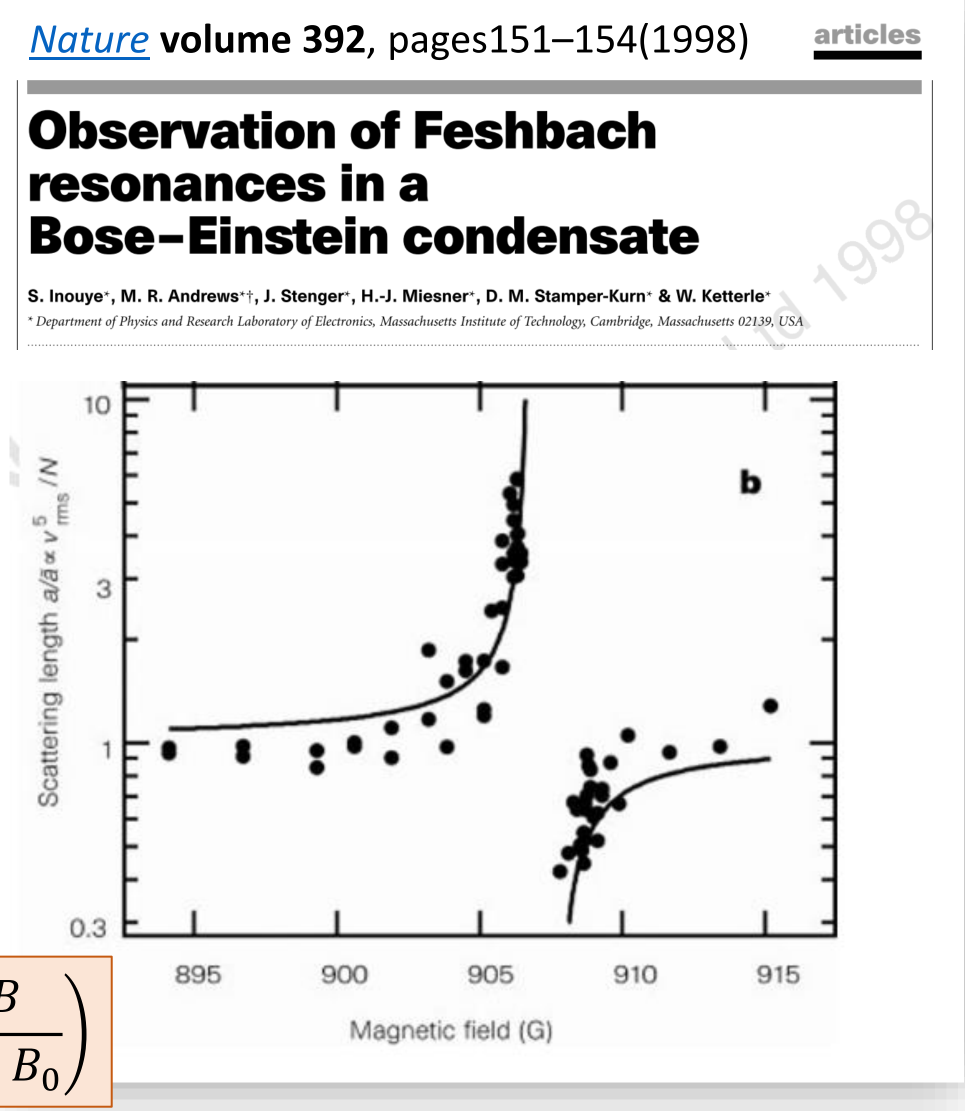

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articles

Observation of Feshbach resonances in a Bose–Einstein condensate

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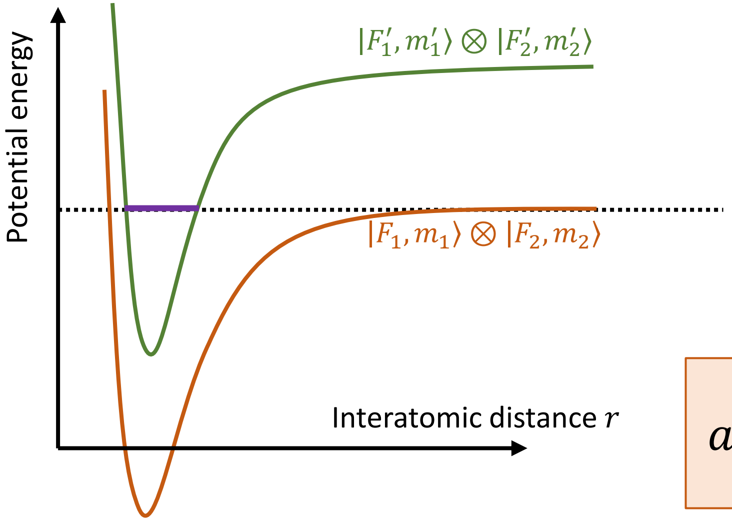
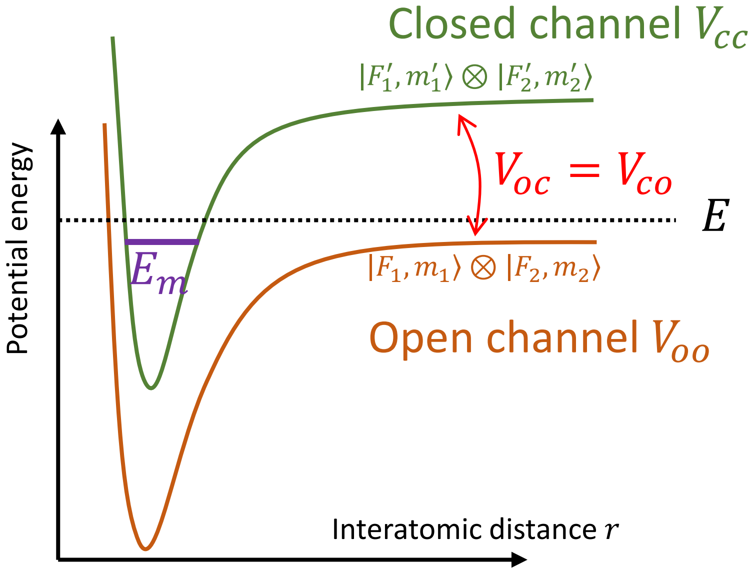


Feshbach resonance
$a \to \infty$

$$a = a_{\rm bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$$

Two-channel model



Radial wave function (for $\ell = 0$):

$$u(r) = u_o(r)|F_1, m_1\rangle \otimes |F_2, m_2\rangle + u_c(r)|F_1', m_1'\rangle \otimes |F_2', m_2'\rangle$$

$$(K + V_{oo} - E)|u_o\rangle + V_{oc}|u_c\rangle = 0$$
$$(K + V_{cc} - E)|u_c\rangle + V_{co}|u_o\rangle = 0$$

With $K = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}$

$$|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|u_c\rangle$$
$$|u_c\rangle = 0 + G_c V_{co}|u_o\rangle$$

With the resolvents:

$G_o = (E - K - V_{oo})^{-1}$

$G_c = (E - K - V_{cc})^{-1}$

And the open-channel eigenstate:

$(K + V_{oo})\,|\bar{u}_o\rangle = E\,|\bar{u}_o\rangle$

$$|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc} G_c V_{co}|u_o\rangle$$

Closed-channel eigenstates:

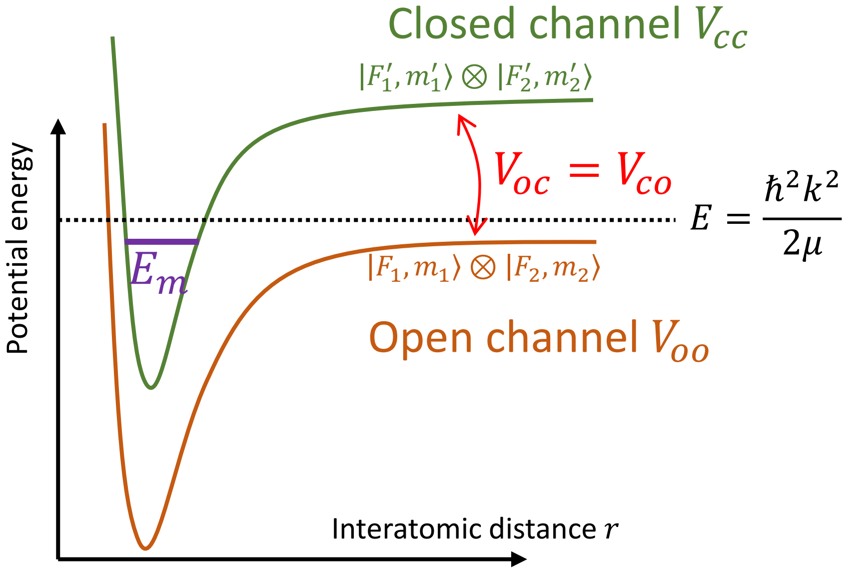
$$(K + V_{cc})\left|\bar{u}_c^{(n)}\right\rangle = E_c^{(n)}\left|\bar{u}_c^{(n)}\right\rangle$$

Isolated resonance approximation:

$$G_c = \sum_n \frac{|\bar{u}_c^{(n)}\rangle\langle\bar{u}_c^{(n)}|}{E - E_c^{(n)}} \approx \frac{|\bar{u}_m\rangle\langle\bar{u}_m|}{E - E_m}$$

$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle\bar{u}_m|V_{co}|u_o\rangle}{E - E_m}}$$

Two-channel model



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$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|u_o\rangle}{E - E_m}}$$

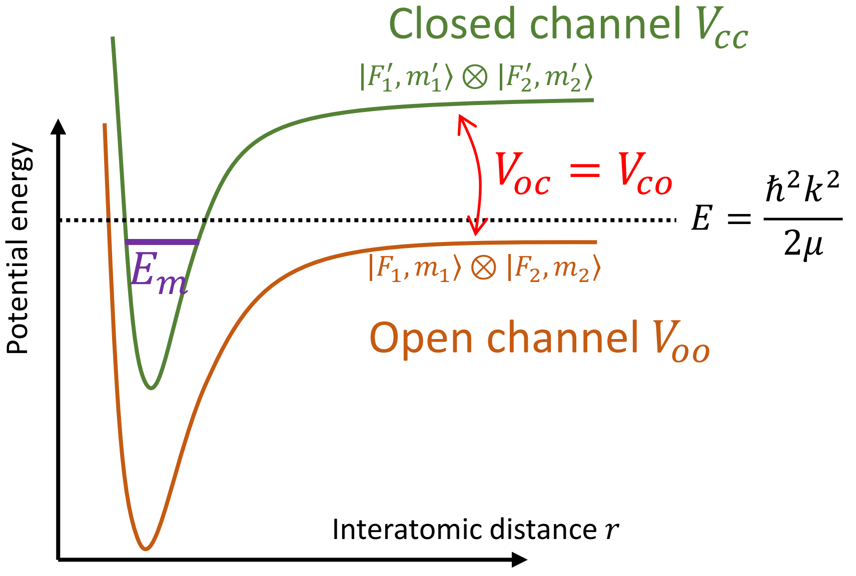
Multiply on the left by $\langle \bar{u}_m|V_{co}$

$$\langle \bar{u}_m|V_{co}|u_o\rangle = \langle \bar{u}_m|V_{co}|\bar{u}_o\rangle + \underbrace{\langle \bar{u}_m|V_{co} G_o V_{oc}|\bar{u}_m\rangle}_{\Delta E_m \text{ : resonance shift}} \frac{\langle \bar{u}_m|V_{co}|u_o\rangle}{E - E_m}$$

$$\langle \bar{u}_m|V_{co}|u_o\rangle = \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{1 - \frac{\Delta E_m}{E - E_m}}$$

$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{E - E_m - \Delta E_m}}$$

Two-channel model



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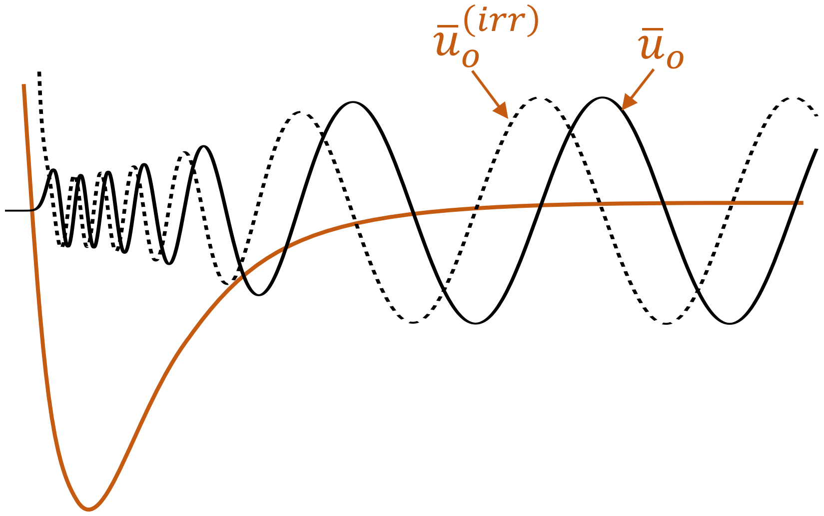
$$\boxed{|u_o\rangle = |\bar{u}_o\rangle + G_o V_{oc}|\bar{u}_m\rangle \frac{\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle}{E - E_m - \Delta E_m}}$$

$$\bar{u}_o(r) \xrightarrow[r \gg b]{} k^{-1} \sin(kr + \bar{\delta}_0)$$

$$\bar{u}_o^{(irr)}(r) \xrightarrow[r \gg b]{} k^{-1} \cos(kr + \bar{\delta}_0)$$

(irregular solution)

$$G_o = -k|\bar{u}_o^{(irr)}\rangle\langle \bar{u}_o|$$

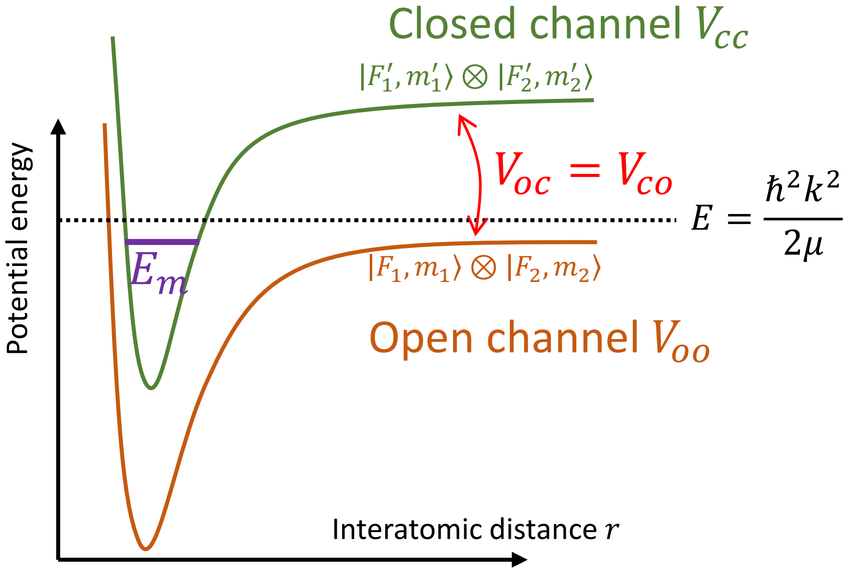


$$|u_o\rangle = |\bar{u}_o\rangle - |\bar{u}_o^{(irr)}\rangle \frac{k\overbrace{|\langle \bar{u}_m|V_{co}|\bar{u}_o\rangle|^2}^{\alpha\text{ : resonance strength}}}{E - E_m - \Delta E_m}$$

$$u_o(r) \xrightarrow[r \gg b]{} k^{-1}\left(\sin(kr + \bar{\delta}_0) - \cos(kr + \bar{\delta}_0)\frac{k\alpha}{E - E_m - \Delta E_m}\right) \propto \sin\left(kr + \underbrace{\bar{\delta}_0 - \arctan\frac{k\alpha}{E - E_m - \Delta E_m}}_{}\right)$$

$$\boxed{\delta_0 = \bar{\delta}_0 - \arctan\frac{k\alpha}{E - E_m - \Delta E_m}}$$

Two-channel model



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$$\boxed{\delta_0 = \bar{\delta}_0 - \arctan \frac{k\alpha}{E - E_m - \Delta E_m}}$$

Background phase shift

Resonant phase shift

For $E \to 0$, $-ka$ $-ka_{\mathrm{bg}}$ Background scattering length

$$a = a_{\mathrm{bg}} - \frac{\alpha}{E_m + \Delta E_m}$$

The scattering length diverges for $E_m + \Delta E_m = 0$

The molecular state energy varies with the magnetic field (approximately linearly)

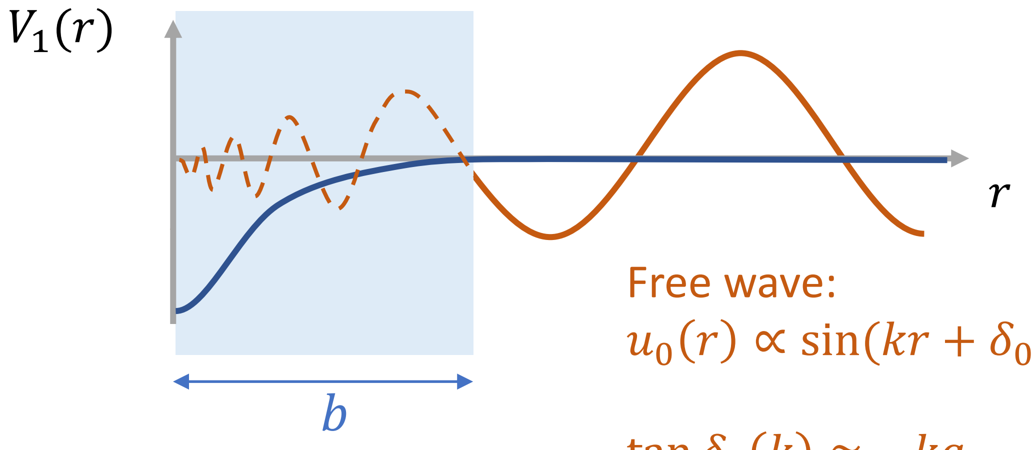
$E_m = \mu_m(B - B_1)$

$$\boxed{a = a_{\mathrm{bg}}\left(1 - \frac{\Delta B}{B - B_0}\right)}$$

The scattering length diverges for $B = B_0 \equiv B_1 - \Delta E_m/\mu_m$

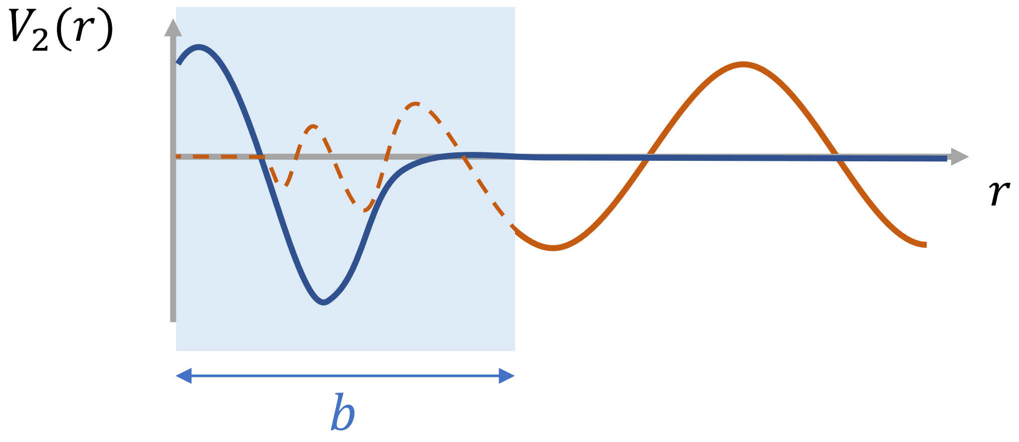
$\Delta B = \dfrac{\alpha}{a_{\mathrm{bg}}\mu_m}$ magnetic width of the resonance

Zero-range theory



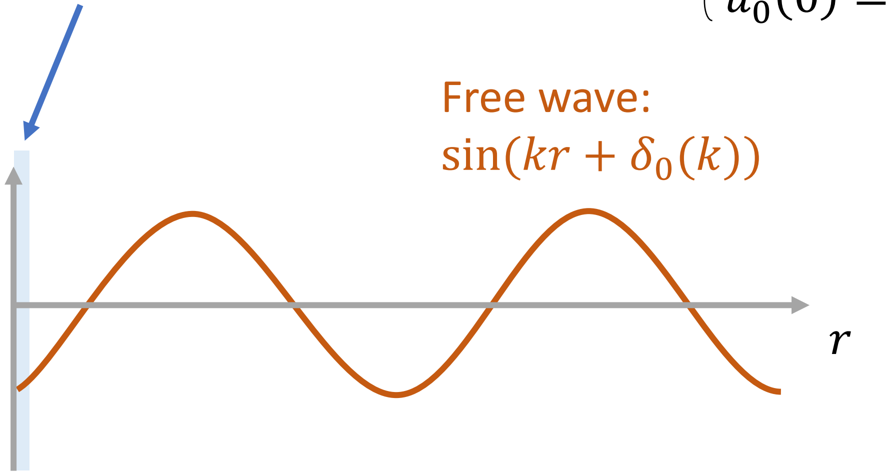
Free wave:
$u_0(r) \propto \sin(kr + \delta_0(k))$

$\tan \delta_0(k) \approx -ka$



Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

Zero-range Boundary condition $\begin{cases} u_0(0) = ? \\ u_0'(0) = ? \end{cases}$



Free wave:
$\sin(kr + \delta_0(k))$

Zero-range theory

Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

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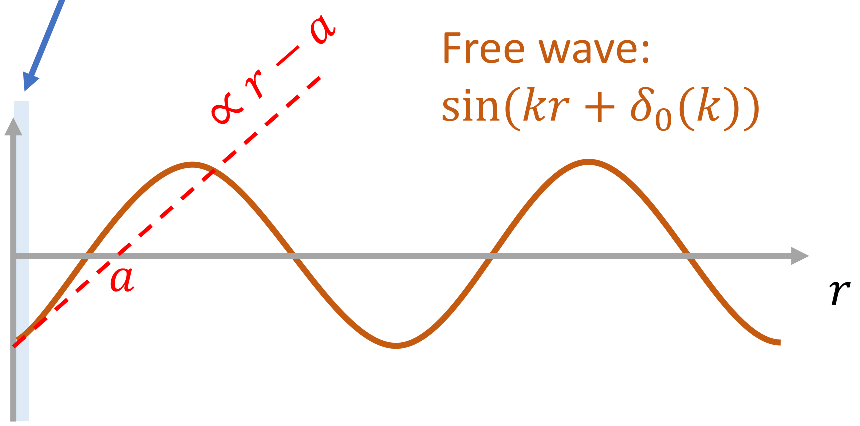


$\tan\delta_0(k) \approx -ka$

Free wave:

$u_0(r) \propto \sin(kr + \delta_0(k))$

$u_0(r) \propto \sin kr + \tan\delta_0(k)\cos kr$

$u_0(r) \propto \sin kr - ka\cos kr$

$u_0'(r) \propto k(\cos kr + ka\sin kr)$

$\begin{cases} u_0(0) \propto -ka \\ u_0'(0) \propto k \end{cases}$ ➡ $\boxed{\frac{u_0'(0)}{u_0(0)} = -\frac{1}{a}}$ →

$u_0(r) \propto kr - ka + O(r^2)$ ➡ $\boxed{u_0(r) \propto r - a}$ →

For $r \to 0$

$\psi(\boldsymbol{r}) = \frac{u_0(r)}{r} + \cdots$

Bethe-Peierls boundary condition:

$$\boxed{\frac{1}{r\psi}\frac{d}{dr}(r\psi) \xrightarrow[r\to 0]{} -\frac{1}{a}}$$

⇕

$$\boxed{\psi \xrightarrow[r\to 0]{} F \times \left(\frac{1}{r} - \frac{1}{a}\right)}$$

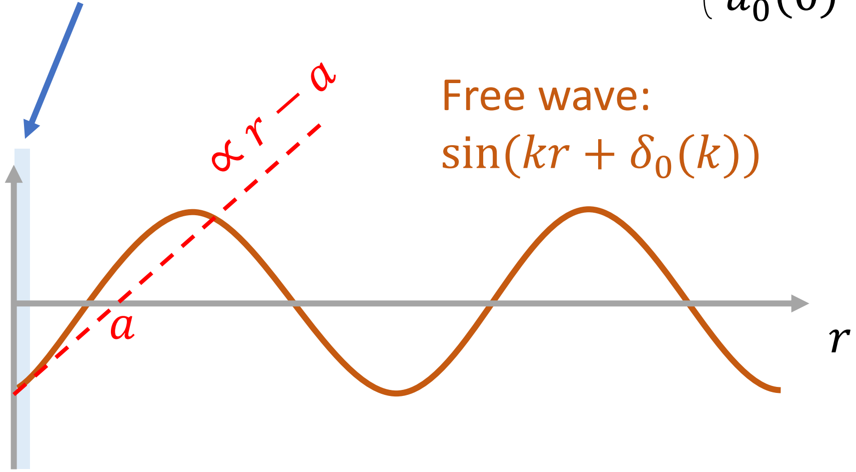
(this condition being isotropic, it only affects the s wave)

➡ **“universal theory” (in terms of a)**

Zero-range theory

Free wave: $-\frac{\hbar^2}{2\mu}\nabla^2\psi = E\psi$

Zero-range Boundary condition $\begin{cases} u_0(0) = ? \\ u_0'(0) = ? \end{cases}$



Bethe-Peierls boundary condition:

$$\boxed{\frac{1}{r\psi}\frac{d}{dr}(r\psi) \underset{r\to 0}{\longrightarrow} -\frac{1}{a}}$$

$$\Updownarrow$$

$$\boxed{\psi \underset{r\to 0}{\longrightarrow} F \times \left(\frac{1}{r} - \frac{1}{a}\right)}$$

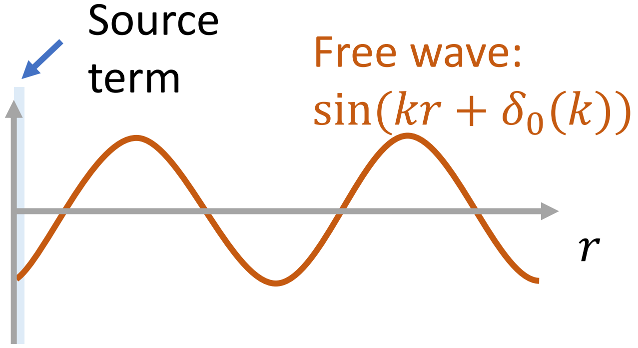
(this condition being isotropic, it only affects the s wave)

⇨ **“universal theory” (in terms of a)**

Alternative formulations:

(1) “source term”

Including the boundary condition by a source term inside the Schrödinger equation:



$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \underbrace{V(r)\psi}_{-\frac{4\pi\hbar^2}{2\mu}F\delta^3(\vec{r})} = E\psi$$

Obtained from the relation: $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$

(2) “Pseudopotential” (Huang-Yang)

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + \underbrace{\frac{4\pi\hbar^2 a}{2\mu}\delta^3(\vec{r})\frac{d}{dr}(r\psi)}_{\text{pseudopotential}} = E\psi$$

(3) “zero-range” or “contact” interaction

Take any potential, set the range to zero, while increasing the depth to keep a fixed scattering length a.

2. Two-body physics 〉 Zero-range theory

(3) “zero-range” or “contact” interaction

Take any potential, set the range to zero, while increasing the depth to keep a fixed scattering length a.

Example in momentum space: $\tilde{V}(\boldsymbol{p}) = \begin{cases} -g & \text{for } p \le b^{-1} \\ 0 & \text{for } p > b^{-1} \end{cases}$

We can set $b \to 0$ only at the end of calculations

$$\frac{\hbar^2 p^2}{2\mu}\tilde{\psi}(\boldsymbol{p}) + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{V}(\boldsymbol{q})\tilde{\psi}(\boldsymbol{p}-\boldsymbol{q}) = E\tilde{\psi}(\boldsymbol{p})$$

At zero energy $E = 0$:

$$p^2\tilde{\psi}(\boldsymbol{p}) - \underbrace{\frac{2\mu}{\hbar^2} g \int_{q<b^{-1}} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{\psi}(\boldsymbol{p}-\boldsymbol{q})}_{f \approx \frac{2\mu}{\hbar^2} g \int_{q<b^{-1}} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\tilde{\psi}(\boldsymbol{q})} = 0$$

$$\tilde{\psi}(\boldsymbol{p}) = (2\pi)^3\delta^3(\boldsymbol{p}) + \frac{f}{p^2}$$

F.T.

$$\tilde{\psi}(\boldsymbol{r}) = 1 + \frac{f}{4\pi r} \equiv 1 - \frac{a}{r} \longrightarrow f = -4\pi a$$

$$f = \frac{2\mu}{\hbar^2} g \left(1 + f \int_{q<b^{-1}} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{q^2}\right)$$

$$\frac{\hbar^2}{2\mu}\frac{1}{g} = \frac{1}{f} + \underbrace{\int_{q<b^{-1}} \frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{q^2}}_{\frac{1}{2\pi^2}b^{-1}}$$

$$\boxed{\frac{4\pi\hbar^2}{2\mu}\frac{1}{g} = -\frac{1}{a} + \frac{2}{\pi}\frac{1}{b}}$$

“Renormalisation relation”

Bethe-Peierls condition in momentum space

Schrödinger equation in coordinate space

$$\boxed{-\frac{\hbar^2}{2\mu}\nabla^2\psi - \frac{4\pi\hbar^2}{2\mu}F\delta^3(\vec{r}) = E\psi}$$

F.T.

Schrödinger equation in momentum space

$$\boxed{\frac{\hbar^2p^2}{2\mu}\tilde{\psi}(\boldsymbol{p}) - \frac{4\pi\hbar^2}{2\mu}F = E\tilde{\psi}(\boldsymbol{p})}$$

$$p^2\tilde{\psi}(\boldsymbol{p}) - 4\pi F = \frac{2\mu}{\hbar^2}E\tilde{\psi}(\boldsymbol{p})$$

$$\lim_{p\to\infty}\left(p^2\tilde{\psi}(\boldsymbol{p}) - 4\pi F\right) = \frac{2\mu}{\hbar^2}E\lim_{p\to\infty}\tilde{\psi}(\boldsymbol{p}) = 0$$

$$4\pi F = \lim_{p\to\infty}p^2\tilde{\psi}(\boldsymbol{p})$$

Condition for the source in coordinate space

$$\boxed{\psi \xrightarrow[r\to 0]{} F\times\left(\frac{1}{r}-\frac{1}{a}\right)}$$

$$\psi(r) - \frac{F}{r} \xrightarrow[r\to 0]{} -F/a$$

$$\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\tilde{\psi}(\boldsymbol{p}) - \frac{4\pi F}{p^2}\right)e^{i\boldsymbol{p}\cdot\boldsymbol{r}} \xrightarrow[r\to 0]{} -F/a$$

$$\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\tilde{\psi}(\boldsymbol{p}) - \frac{4\pi F}{p^2}\right) = -F/a$$

Condition for the source in momentum space

$$\boxed{F = -a\int\frac{d^3p}{(2\pi)^3}\left(\tilde{\psi}(\boldsymbol{p}) - \frac{\lim_{q\to\infty}q^2\tilde{\psi}(\boldsymbol{q})}{p^2}\right)}$$

Bound state in the Zero-Range Theory

There is only one bound state in the zero-range model. It has zero angular momentum.

We look for eigenstates of energy $E = -\hbar^2\kappa^2/2\mu$

Coordinate space

Equation: $-\dfrac{d^2u(r)}{dr^2} = -\kappa^2 u(r)$

$$u(r) = Ae^{-\kappa r} + \cancel{Be^{+\kappa r}}$$

Condition: $\dfrac{du(r)}{dr} = -\dfrac{1}{a}u(r)$

$$-\kappa u(r) = -\frac{1}{a}u(r)$$

➡ $\boxed{\kappa = \dfrac{1}{a}}$

$$\psi(\boldsymbol{r}) = \frac{e^{-r/a}}{\sqrt{2\pi a}r}$$

Momentum space

Equation: $p^2\,\tilde{\psi}(\boldsymbol{p}) - 4\pi F = -\kappa^2\tilde{\psi}(\boldsymbol{p})$

$$\tilde{\psi}(\boldsymbol{p}) = \frac{4\pi F}{p^2 + \kappa^2}$$

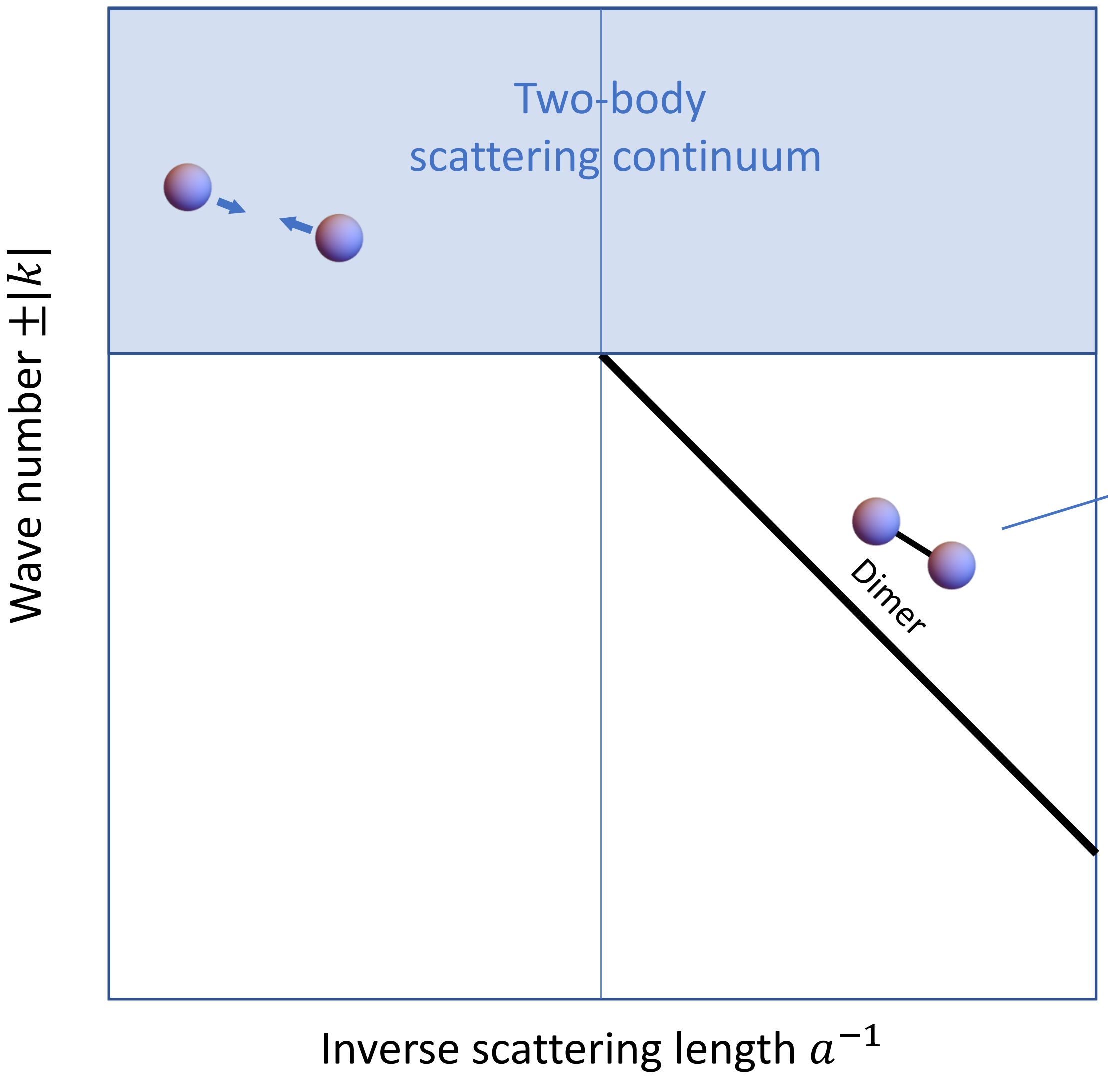
Condition: $\int \dfrac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\tilde{\psi}(\boldsymbol{p}) - \dfrac{4\pi F}{p^2}\right) = -F/a$

$$F\,\underbrace{4\pi\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left(\frac{1}{p^2+\kappa^2} - \frac{1}{p^2}\right)}_{-\kappa} = -F/a$$

➡ $\boxed{\kappa = \dfrac{1}{a}}$

$$\tilde{\psi}(\boldsymbol{p}) = \frac{1}{\pi\sqrt{a}}\frac{1}{p^2 + a^{-2}}$$

Two-body spectrum in the Zero-Range Theory



$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{2\pi a r}} e^{-r/a}$$

$$E = -\frac{\hbar^2}{2\mu}\kappa^2$$

With
$\kappa = a^{-1}$

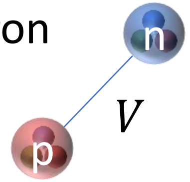
Three-body physics

The Thomas collapse (1935)

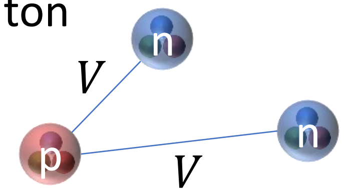
Llewellyn Thomas in 1926

“The Interaction Between a Neutron and a Proton and the Structure of H3.”, Phys. Rev. 47, 903, 1935

deuteron



triton



$$H_{2B} = T_1 + T_2 + V(r_{12})$$

$$H_{3B} = T_1 + T_2 + T_3 + V(r_{12}) + V(r_{13})$$

Ground energy: E_{2B}

Ground energy: $E_{3B} \leq \frac{\langle\psi|H_{3B}|\psi\rangle}{\langle\psi|\psi\rangle}$

$$\frac{\langle\psi|H_{3B}|\psi\rangle}{\langle\psi|\psi\rangle} = -\frac{constant}{b^2}|E_{2B}|$$

For a particular ansatz ψ

Collapse:

$E_{3B} \rightarrow -\infty$ when $b \rightarrow 0$
Why? *This was a mystery*.

Estimate of b:

From the known ratio $\frac{E_{3B}}{E_{2B}} = 4$,

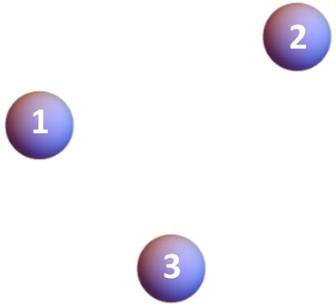
Thomas found $b \sim 6 \cdot 10^{-15}$m

The Skorniakov – Ter-Martirosian equation (1955)

Karen Avetikovich Ter-Martirosian
(undated)

G. Skorniakov and K. Ter-Martirosian, "Three Body Problem for Short Range Forces. I. Scattering of Low Energy Neutrons by Deuterons," **Sov. Phys. JETP, 4, 648, 1957**.

General three-body equation

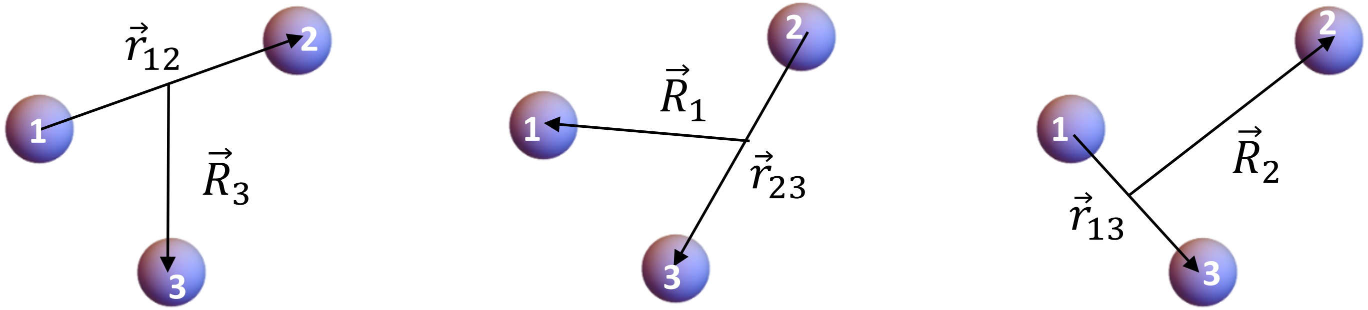


Three-body wave function: $\Psi(\vec{x}_1, \vec{x}_2, \vec{x}_3)$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$

1. Eliminate the centre of mass $\vec{R} = (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)/3$

2. Express the remaining coordinates in terms of Jacobi coordinates: $\Psi(\vec{R}, \vec{r})$

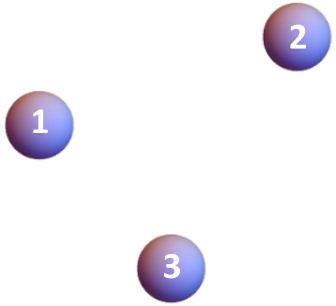


$$\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$$

$$\vec{R}_k = \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2}$$

$$\hat{H} = -\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$

General three-body equation



$$\hat{H} = -\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23})$$

Schrödinger equation at energy E

$$\left(-\frac{3\hbar^2}{4m}\nabla_R^2 - \frac{\hbar^2}{m}\nabla_r^2 + V(r_{12}) + V(r_{13}) + V(r_{23}) - E\right)\Psi(\vec{R},\vec{r}) = 0$$

Schrödinger equation at energy $E = -\frac{\hbar^2\kappa^2}{m} < 0$

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi\left(\vec{R},\vec{r}\right) \quad = -\frac{m}{\hbar^2}\big(V(r_{12}) + V(r_{13}) + V(r_{23})\big)\Psi(\vec{R},\vec{r})$$

Zero-range limit

Schrödinger equation at energy $E = -\frac{\hbar^2\kappa^2}{m} < 0$

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi\left(\vec{R},\vec{r}\right) \;= -\frac{m}{\hbar^2}\left(V(r_{12}) + V(r_{13}) + V(r_{23})\right)\Psi(\vec{R},\vec{r})$$

Zero-range limit

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi\left(\vec{R},\vec{r}\right) = 4\pi[F_1(R_1)\delta^3(r_{23}) + F_2(R_2)\delta^3(r_{13}) + F_3(R_3)\delta^3(r_{12})]$$

with $\Psi\left(\vec{R}_k,\vec{r}_{ij}\right) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right) F_k(\vec{R}_k)$

(Bethe-Peierls condition)

The Skorniakov & Ter-Martirosian equation

$$\left(-\frac{3}{4}\nabla_R^2 - \nabla_r^2 + \kappa^2\right)\Psi\left(\vec{R},\vec{r}\right) = 4\pi[F_1(R_1)\delta^3(r_{23}) + F_2(R_2)\delta^3(r_{13}) + F_3(R_3)\delta^3(r_{12})]$$

In momentum (Fourier) representation

with $\Psi\left(\vec{R}_k,\vec{r}_{ij}\right) \xrightarrow[r_{ij}\to 0]{} \left(\frac{1}{r_{ij}} - \frac{1}{a_{ij}}\right)F_k(\vec{R}_k)$ ②

$$\left(\frac{3}{4}P^2 + p^2 + \kappa^2\right)\tilde{\Psi}\left(\vec{P},\vec{p}\right) = 4\pi\sum_{i=1,2,3}\tilde{F}_i(\vec{P}_i)$$

$$\tilde{\Psi}\left(\vec{P},\vec{p}\right) = \frac{4\pi\sum_{i=1,2,3}\tilde{F}_i\left(\vec{P}_i\right)}{\frac{3}{4}P^2 + p^2 + \kappa^2}$$ ①

Skorniakov – Ter-Martirosian integral equations:

$$\boxed{\left(\frac{1}{a_{ij}} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}_k\left(\vec{P}\right) + 4\pi\int\frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}_i\left(\vec{Q}\right) + \tilde{F}_j\left(\vec{Q}\right)}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

with $\{i,j,k\} = \{1,2,3\}$

Benefit of the zero-range theory:

Now, the unknown function has only 1 argument!

$\tilde{\Psi}\left(\vec{P},\vec{p}\right) \longrightarrow \tilde{F}\left(\vec{P}\right)$

The Skorniakov & Ter-Martirosian equation

Detailed derivation (1/3):

$$\tilde{\Psi}\left(\vec{P},\vec{p}\right) = \frac{4\pi\sum_{i=1,2,3}\tilde{F}_i\left(\vec{P}_i\right)}{\frac{3}{4}P^2+p^2+\kappa^2}$$ ①

with $\Psi\left(\vec{R}_k,\vec{r}_{ij}\right)\xrightarrow[r_{ij}\to 0]{}\left(\frac{1}{r_{ij}}-\frac{1}{a_{ij}}\right)F_k(\vec{R}_k)$ ②

Consider the function $\Omega\left(\vec{R}_k,\vec{r}_{ij}\right)=\Psi\left(\vec{R}_k,\vec{r}_{ij}\right)-\frac{1}{r_{ij}}F_k(\vec{R}_k)$, which removes the $1/r_{ij}$ divergence from Ψ.

According to ② , this function goes to the finite value $-1/a_{ij}F_k(\vec{R}_k)$ when $r_{ij}\to 0$.
Let us consider its Fourier transform: $\tilde{\Omega}\left(\vec{P}_k,\vec{p}_{ij}\right)=\tilde{\Psi}\left(\vec{P}_k,\vec{p}_{ij}\right)-\frac{4\pi}{p_{ij}^2}\tilde{F}_k(\vec{P}_k)$.

We have $\Omega\left(\vec{R}_k,\vec{r}_{ij}\right)=\int\frac{d^3\vec{P}}{(2\pi)^3}\int\frac{d^3\vec{p}}{(2\pi)^3}\Omega\left(\vec{P},\vec{p}\right)e^{i\left(\vec{P}\cdot\vec{R}_k+\vec{p}\cdot\vec{r}_{ij}\right)}$, so $\Omega\left(\vec{R}_k,\vec{0}\right)=\int\frac{d^3\vec{P}}{(2\pi)^3}\int\frac{d^3\vec{p}}{(2\pi)^3}\Omega\left(\vec{P},\vec{p}\right)e^{i\vec{P}\cdot\vec{R}_k}$, which is equal to$-1/a_{ij}F_k(\vec{R}_k)$. Taking the Fourier transform again, we arrive at $\int\frac{d^3\vec{p}}{(2\pi)^3}\Omega\left(\vec{P},\vec{p}\right)=-1/a_{ij}\tilde{F}_k(\vec{P})$

i.e.

$$\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\tilde{\Psi}\left(\vec{P}_k,\vec{p}_{ij}\right)-\frac{4\pi}{p_{ij}^2}\tilde{F}_k\left(\vec{P}_k\right)\right]=-\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$ ③

The Skorniakov & Ter-Martirosian equation

Detailed derivation (2/3):

$$\widetilde{\Psi}\left(\vec{P},\vec{p}\right)=\frac{4\pi\sum_{i=1,2,3}\tilde{F}_i\left(\vec{P}_i\right)}{\frac{3}{4}P^2+p^2+\kappa^2}$$ ①

with $$\Psi\left(\vec{R}_k,\vec{r}_{ij}\right)\xrightarrow[r_{ij}\to 0]{}\left(\frac{1}{r_{ij}}-\frac{1}{a_{ij}}\right)F_k(\vec{R}_k)$$ ②

$$\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\widetilde{\Psi}\left(\vec{P}_k,\vec{p}_{ij}\right)-\frac{4\pi}{p_{ij}^2}\tilde{F}_k\left(\vec{P}_k\right)\right]=-\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$ ③

$$4\pi\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\frac{\tilde{F}_i\left(\vec{P}_i\right)+\tilde{F}_j\left(\vec{P}_j\right)+\tilde{F}_k\left(\vec{P}_k\right)}{\frac{3}{4}P_k^2+p_{ij}^2+\kappa^2}-\frac{1}{p_{ij}^2}\tilde{F}_k\left(\vec{P}_k\right)\right]=-\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$

$$\underbrace{4\pi\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\left[\frac{1}{p_{ij}^2+\left(\frac{3}{4}P_k^2+\kappa^2\right)}-\frac{1}{p_{ij}^2}\right]}_{-\sqrt{\frac{3}{4}P_k^2+\kappa^2}}\tilde{F}_k\left(\vec{P}_k\right)+4\pi\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\frac{\tilde{F}_i\left(\vec{P}_i\right)+\tilde{F}_j\left(\vec{P}_j\right)}{\frac{3}{4}P_k^2+p_{ij}^2+\kappa^2}=-\frac{1}{a_{ij}}\tilde{F}_k(\vec{P}_k)$$

The Skorniakov & Ter-Martirosian equation

Detailed derivation (3/3):

$$\left(\frac{1}{a_{ij}}-\sqrt{\frac{3}{4}P_k^2+\kappa^2}\right)\tilde{F}_k\left(\vec{P}_k\right)+4\pi\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\frac{\tilde{F}_i\left(\vec{P}_i\right)}{\frac{3}{4}P_k^2+p_{ij}^2+\kappa^2}+4\pi\int\frac{d^3\vec{p}_{ij}}{(2\pi)^3}\frac{\tilde{F}_j\left(\vec{P}_j\right)}{\frac{3}{4}P_k^2+p_{ij}^2+\kappa^2}=0$$

Using $\vec{p}_{ij}=-\vec{P}_i-\frac{1}{2}\vec{P}_k$ and $\vec{p}_{ij}=\vec{P}_j+\frac{1}{2}\vec{P}_k$ to make a change of integration variable $\vec{p}_{ij}\rightarrow\vec{P}_i$ and $\vec{p}_{ij}\rightarrow\vec{P}_j$ in the two integrals, one obtains:

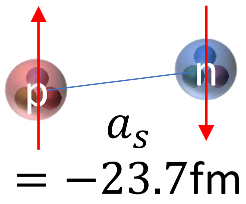
$$\left(\frac{1}{a_{ij}}-\sqrt{\frac{3}{4}P_k^2+\kappa^2}\right)\tilde{F}_k\left(\vec{P}_k\right)+4\pi\int\frac{d^3\vec{P}_i}{(2\pi)^3}\frac{\tilde{F}_i\left(\vec{P}_i\right)}{\frac{3}{4}P_k^2+\left(\vec{P}_i+\frac{1}{2}\vec{P}_k\right)^2+\kappa^2}+4\pi\int\frac{d^3\vec{P}_j}{(2\pi)^3}\frac{\tilde{F}_j\left(\vec{P}_j\right)}{\frac{3}{4}P_k^2+\left(\vec{P}_j+\frac{1}{2}\vec{P}_k\right)^2+\kappa^2}=0$$

Finally, relabelling the integration variables $\vec{P}_i$ and $\vec{P}_j$ as $\vec{Q}$, one arrives at the Skorniakov – Ter-Martirosian equations:

$$\boxed{\left(\frac{1}{a_{ij}}-\sqrt{\frac{3}{4}P^2+\kappa^2}\right)\tilde{F}_k\left(\vec{P}\right)+4\pi\int\frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}_i\left(\vec{Q}\right)+\tilde{F}_j\left(\vec{Q}\right)}{P^2+Q^2+\vec{Q}\cdot\vec{P}+\kappa^2}=0}$$

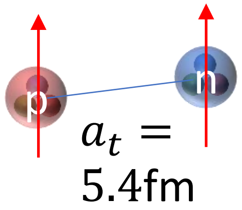
Application to nucleons:

Nucleon interaction:



Singlet $S = 0$

$|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle$



Triplet $S = 1$

$|\uparrow\rangle|\uparrow\rangle$

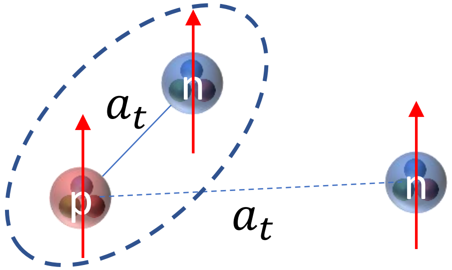
$|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle$

$|\downarrow\rangle|\downarrow\rangle$

Deuteron-neutron scattering:

deuteron ($s' = 1$)   neutron (s= 1/2)

$S = 3/2$



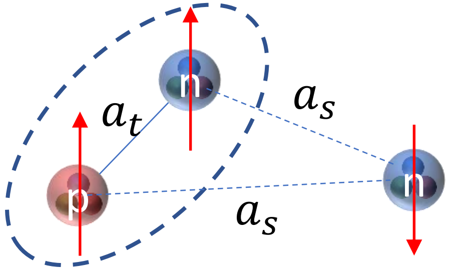
$F_3 = -F_2 \equiv F \qquad F_1 = 0$

$$\boxed{\left(\frac{1}{a_t} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}\left(\vec{P}\right) - 4\pi \int \frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}\left(\vec{Q}\right)}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

Neutron-deuteron scatt. Length: $\boxed{a_{nd} = 1.18 a_t}$

Universal result

$S = 1/2$



Simplified version: $a_t = a_s \qquad F_1 = F_2 = F_3 \equiv F$

(equivalent to the problem of 3 identical bosons)

$$\boxed{\left(\frac{1}{a_t} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}\left(\vec{P}\right) + 8\pi \int \frac{d^3\vec{Q}}{(2\pi)^3}\frac{\tilde{F}\left(\vec{Q}\right)}{P^2 + Q^2 + \vec{Q}\cdot\vec{P} + \kappa^2} = 0}$$

Problem: energy is not bound from below!

The analytical solution of Minlos-Faddeev (1961)

Skorniakov – Ter-Martirosian integral equation for three bosons at unitarity ($a \to \infty$):

$$\left(\cancel{\frac{1}{a}} - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(P) + \frac{2}{\pi}\int_0^\infty \frac{Q}{P}dQ \ln\frac{Q^2 + P^2 + PQ + \kappa^2}{Q^2 + P^2 - PQ + \kappa^2}\tilde{F}(Q) = 0$$

1. Extension of integration to $[-\infty, \infty]$

$$\left(0 - \sqrt{\frac{3}{4}P^2 + \kappa^2}\right)\tilde{F}(P) + \frac{1}{\pi}\int_{-\infty}^\infty \frac{Q}{P}dQ \ln\frac{Q^2 + P^2 + PQ + \kappa^2}{Q^2 + P^2 - PQ + \kappa^2}\tilde{F}(Q) = 0$$

2. Change of variables: $P = \frac{1}{\sqrt{3}}\kappa\left(z - \frac{1}{z}\right)$ $\qquad Q = \frac{1}{\sqrt{3}}\kappa(z' - 1/z')$

$$-g(z) + \frac{4}{\pi\sqrt{3}}\int_0^\infty \frac{dz'}{z'}\ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right)g(z') = 0$$

where $g(z) \equiv \left(z^2 - \frac{1}{z^2}\right)\tilde{F}(P)$

The analytical solution of Minlos-Faddeev (1961)

$$-g(z) + \frac{4}{\pi\sqrt{3}} \int_0^\infty \frac{dz'}{z'} \ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right) g(z') = 0$$

where $g(z) \equiv \left(z^2 - \frac{1}{z^2}\right) F(P)$

Scale invariance: $z \rightarrow \lambda z$ (if $g(z)$ is a solution, then $g(\lambda z)$ is also a solution)

3. Solution of the form: $g(z) = z^s$

$$-z^s + \frac{4}{\pi\sqrt{3}} \int_0^\infty dz' \ln\left(\frac{z'^2 + z^2 + zz'}{z'^2 + z^2 - zz'}\right) z'^{s-1} = 0$$

$$-1 + \frac{4}{\pi\sqrt{3}} \underbrace{\int_0^\infty dx' \ln\left(\frac{1 + x^2 + x}{1 + x^2 - x}\right) x^{s-1}}_{\frac{2\pi}{s}\frac{\sin\left(\frac{\pi}{6}s\right)}{\cos\left(\frac{\pi}{2}s\right)}} = 0$$

for $-1 < \mathrm{Re}(s) < 1$

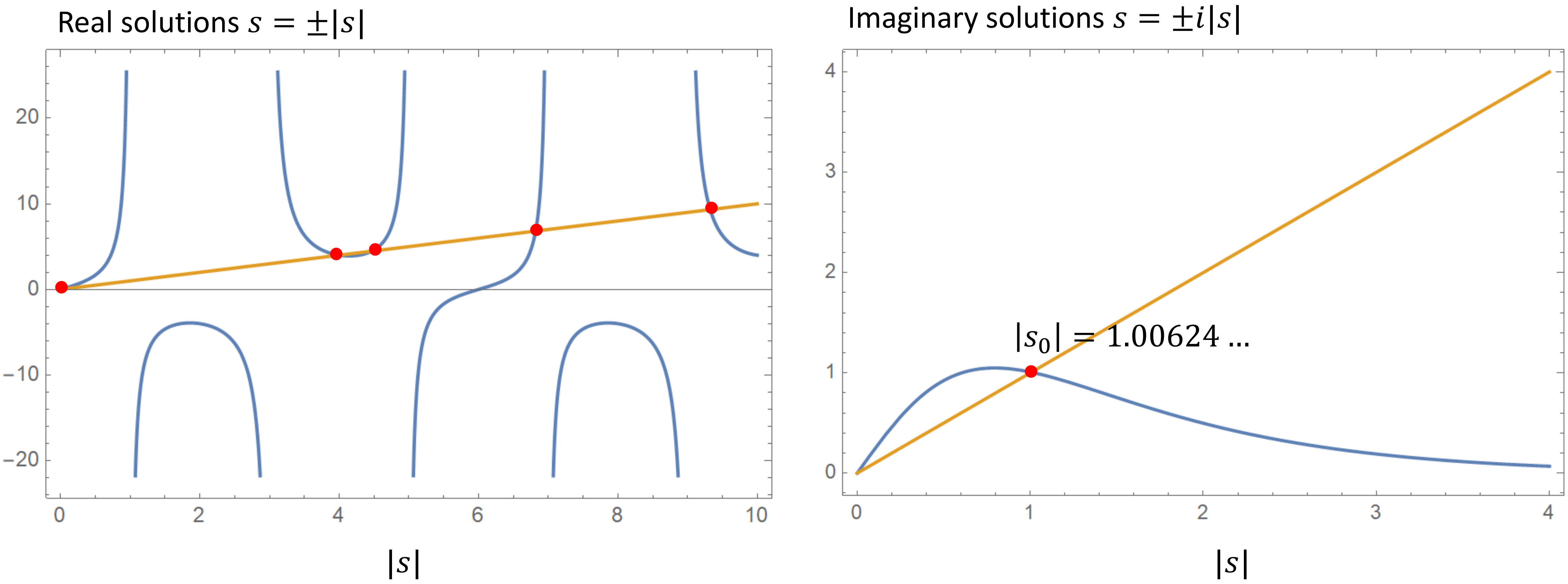
Transcendental equation:

$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6}s\right)}{\sqrt{3}\cos\left(\frac{\pi}{2}s\right)}}$$

3. Three-body physics 〉 Skorniakov – Ter-Martirosian's theory (1955)

3. Solution of the form: $g(z) = z^s$

$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6} s\right)}{\sqrt{3} \cos\left(\frac{\pi}{2} s\right)}}$$



3. Solution of the form: $g(z) = z^s$

$$\boxed{s = \frac{8\pi \sin\left(\frac{\pi}{6}s\right)}{\sqrt{3}\cos\left(\frac{\pi}{2}s\right)}}$$

$$g(z) = C_+ z^{i|s_0|} + C_- z^{-i|s_0|}$$

Since $g(z) = \left(z^2 - z^{-2}\right)\tilde{F}(P)$, we have $g(1) = 0$, therefore $C_+ = -C_- \equiv C$.

$$g(z) = C\left(z^{i|s_0|} - z^{-i|s_0|}\right)$$

4. Going back to the original variables, we obtain the solution:

$$\boxed{\tilde{F}(P) \propto \frac{1}{P\sqrt{1+\frac{3P^2}{4\kappa^2}}}\sin\left(|s_0|\operatorname{arcsinh}\frac{\sqrt{3}P}{2\kappa}\right)}$$

Solution valid for any κ, i.e. any negative energy!
(consistent with Thomas collapse)

Obviously something is wrong

Vitaly Efimov in 1977

Efimov's breakthrough (1970)

V. Efimov, "Weakly-bound states of three resonantly-interacting particles," **Yad. Fiz., 12, 1080–1091**, November 1970, [**Sov. J. Nucl. Phys. 12, 589-595 (1971)**].

V. Efimov, "Energy levels arising from resonant two-body forces in a three-body system." **Physics Letters B, 33, 563 – 564, 1970**.

Derivation for three identical bosons

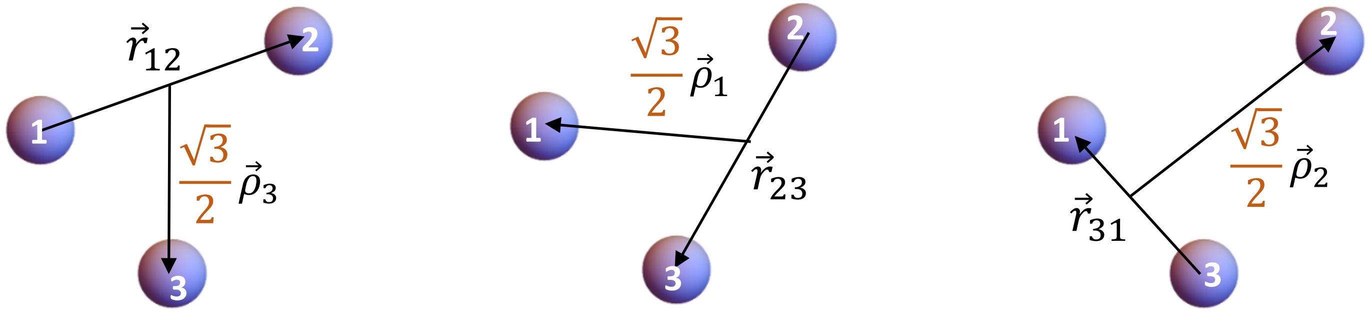
Hamiltonian in coordinate representation:

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 - \frac{\hbar^2}{2m}\nabla_3^2$$

with the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$

1. Eliminate the centre of mass $\vec{R} = \vec{x}_1 + \vec{x}_2 + \vec{x}_3$

2. Express the remaining coordinates in terms of Jacobi coordinates:



$$\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$$

$$\frac{\sqrt{3}}{2}\vec{\rho}_k = \vec{x}_k - \frac{\vec{x}_i + \vec{x}_j}{2}$$

Schrödinger equation:

$$\left(-\nabla_{r_{12}}^2 - \nabla_{\rho_3}^2 - k^2\right)\Psi = 0$$

For a total energy $E = \hbar^2 k^2/m$

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec{r}_{12}, \vec{\rho}_3) + \chi(\vec{r}_{23}, \vec{\rho}_1) + \chi(\vec{r}_{31}, \vec{\rho}_2)$$

$$= \chi(\vec{r}, \vec{\rho}) + \chi\left(-\tfrac{1}{2}\vec{r} + \tfrac{\sqrt{3}}{2}\vec{\rho}, -\tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right) + \chi\left(-\tfrac{1}{2}\vec{r} - \tfrac{\sqrt{3}}{2}\vec{\rho}, \tfrac{\sqrt{3}}{2}\vec{r} - \tfrac{1}{2}\vec{\rho}\right)$$

Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec{r}, \vec{\rho}) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij} \to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ $\Longleftrightarrow$ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec r_{12},\vec\rho_3) + \chi(\vec r_{23},\vec\rho_1) + \chi(\vec r_{31},\vec\rho_2)$$
$$= \chi(\vec r,\vec\rho) + \chi\left(-\tfrac12\vec r + \tfrac{\sqrt3}{2}\vec\rho, -\tfrac{\sqrt3}{2}\vec r - \tfrac12\vec\rho\right) + \chi\left(-\tfrac12\vec r - \tfrac{\sqrt3}{2}\vec\rho, \tfrac{\sqrt3}{2}\vec r - \tfrac12\vec\rho\right)$$

Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec r,\vec\rho) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ ⟺ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec r,\vec\rho)\right)\right]_{r\to0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac12\vec r} + \tfrac{\sqrt3}{2}\vec\rho, -\cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac12\vec r} - \tfrac{\sqrt3}{2}\vec\rho, \cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0}$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec r,\vec\rho) + \chi\left(-\cancel{\tfrac12\vec r} + \tfrac{\sqrt3}{2}\vec\rho, -\cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right) + \chi\left(-\cancel{\tfrac12\vec r} - \tfrac{\sqrt3}{2}\vec\rho, \cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0}$$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec r,\vec\rho)\right)\right]_{r\to0} + \chi\left(\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right) + \chi\left(-\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right)$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec r,\vec\rho) + \cancel{\chi\left(\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right)} + \cancel{\chi\left(-\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right)}\right)\right]_{r\to0}$$

Derivation for three identical bosons

3. Make the Faddeev decomposition:

$$\Psi = \chi(\vec r_{12},\vec\rho_3) + \chi(\vec r_{23},\vec\rho_1) + \chi(\vec r_{31},\vec\rho_2)$$

$$= \chi(\vec r,\vec\rho) + \chi\left(-\tfrac12\vec r + \tfrac{\sqrt3}{2}\vec\rho, -\tfrac{\sqrt3}{2}\vec r - \tfrac12\vec\rho\right) + \chi\left(-\tfrac12\vec r - \tfrac{\sqrt3}{2}\vec\rho, \tfrac{\sqrt3}{2}\vec r - \tfrac12\vec\rho\right)$$

Where χ satisfies

$$\left(-\nabla_r^2 - \nabla_\rho^2 - k^2\right)\chi(\vec r,\vec\rho) = 0$$

4. Apply the two-body condition $\Psi \xrightarrow[r_{ij}\to 0]{} \propto \frac{1}{r_{ij}} - \frac{1}{a}$ $\Longleftrightarrow$ $\frac{\partial}{\partial r}(r\Psi) = -\frac{1}{a}(r\Psi)$ for $r \to 0$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec r,\vec\rho)\right)\right]_{r\to0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac12\vec r} + \tfrac{\sqrt3}{2}\vec\rho, -\cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0} + \left[\frac{\partial}{\partial r}\left(r\chi\left(-\cancel{\tfrac12\vec r} - \tfrac{\sqrt3}{2}\vec\rho, \cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0}$$

$$= -\frac{1}{a}\left[r\left(\chi(\vec r,\vec\rho) + \chi\left(-\cancel{\tfrac12\vec r} + \tfrac{\sqrt3}{2}\vec\rho, -\cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right) + \chi\left(-\cancel{\tfrac12\vec r} - \tfrac{\sqrt3}{2}\vec\rho, \cancel{\tfrac{\sqrt3}{2}\vec r} - \tfrac12\vec\rho\right)\right)\right]_{r\to0}$$

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec r,\vec\rho)\right)\right]_{r\to0} + \chi\left(\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right) + \chi\left(-\tfrac{\sqrt3}{2}\vec\rho, -\tfrac12\vec\rho\right)$$

$$= -\frac{1}{a}\left[r\chi(\vec r,\vec\rho)\right]_{r\to0}$$

Derivation for three identical bosons

Equation:

$$\left(-\nabla_{r}^{2}-\nabla_{\rho}^{2}-k^{2}\right)\chi(\vec{r},\vec{\rho})=0$$

Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\left(r\chi(\vec{r},\vec{\rho})\right)\right]_{r\to 0}+\chi\left(\frac{\sqrt{3}}{2}\vec{\rho},-\frac{1}{2}\vec{\rho}\right)+\chi\left(-\frac{\sqrt{3}}{2}\vec{\rho},-\frac{1}{2}\vec{\rho}\right)=-\frac{1}{a}\left[r\chi(\vec{r},\vec{\rho})\right]_{r\to 0}$$

5. Expand χ in partial waves. For a total angular momentum $L=0$,

$$\chi(\vec{r},\vec{\rho})=\frac{\chi_0(r,\rho)}{r\rho}$$

$\Rightarrow$ $\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r-\frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}\rho-k^2\right)\frac{\chi_0(r,\rho)}{r\rho}=0$ with $\left[\frac{\partial}{\partial r}\frac{\chi_0(r,\rho)}{\rho}\right]_{r\to 0}+2\times\frac{\chi_0\left(\frac{\sqrt{3}}{2}\rho,\frac{1}{2}\rho\right)}{\frac{\sqrt{3}}{2}\rho\cdot\frac{1}{2}\rho}=-\frac{1}{a}\frac{\chi_0(0,\rho)}{\rho}$

$\Rightarrow$ $\left(-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial \rho^2}-k^2\right)\chi_0(r,\rho)=0$ with $\left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0}+\frac{8}{\sqrt{3}\rho}\chi_0\left(\frac{\sqrt{3}}{2}\rho,\frac{1}{2}\rho\right)=-\frac{1}{a}\chi_0(0,\rho)$

Derivation for three identical bosons

Equation:

$$\left(-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial \rho^2}-k^2\right)\chi_0(r,\rho)=0$$

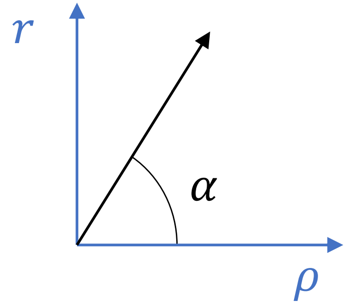
Boundary condition $r \to 0$:

$$\left[\frac{\partial}{\partial r}\chi_0(r,\rho)\right]_{r\to 0}+\frac{8}{\sqrt{3}\rho}\chi_0\left(\tfrac{\sqrt{3}}{2}\rho,\tfrac{1}{2}\rho\right)=-\frac{1}{a}\chi_0(0,\rho)$$

6. Change the coordinates (r,ρ) to polar coordinates (R,α)

$r = R\sin\alpha$ $\quad$ $R=\sqrt{r^2+\rho^2}$ (hyper-radius)

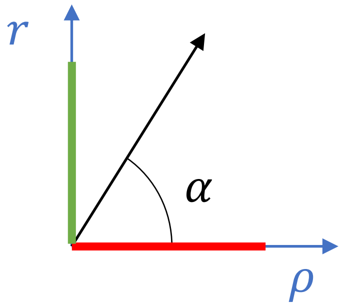
$\rho = R\cos\alpha$ $\quad$ $\alpha = \arctan r/\rho$ (hyper-angle)



$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

with $\left[\frac{\partial}{\partial\alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$

Derivation for three identical bosons



Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial\alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

Boundary condition $\alpha\to 0$:

$$\left[\frac{\partial}{\partial\alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$$

Boundary condition $\alpha\to\frac{\pi}{2}$: $\chi_0\left(R,\frac{\pi}{2}\right)=0$

$$\chi(\vec{r},\vec{\rho})=\frac{\chi_0(r,\rho)}{r\rho}\quad\Longrightarrow[\chi_0(r,\rho)]_{\rho\to 0}=0$$

7. **At unitarity** $a\to\infty$, the two boundary conditions are independent of R.
Therefore, the problem becomes separable in R and α

Solutions of the form: $\chi_0(R,\alpha)=F_n(R)\phi_n(\alpha)$

Boundary condition $\alpha\to\frac{\pi}{2}$: OK

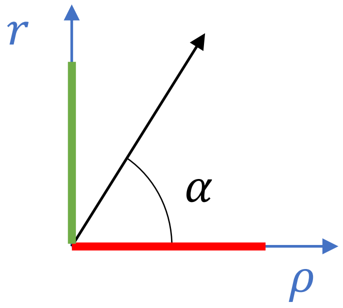
Eigenfunctions of $-\partial^2/\partial\alpha^2$:

$$-\frac{\partial^2}{\partial\alpha^2}\phi_n(\alpha)=s_n^2\phi_n(\alpha)\quad ➡ \quad\phi_n(\alpha)=\sin\left(s_n\left(\frac{\pi}{2}-\alpha\right)\right)$$

Boundary condition $\alpha\to 0$:

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

Derivation for three identical bosons



Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}-k^2\right)\chi_0(R,\alpha)=0$$

Solutions of the form: $\chi_0(R,\alpha)=F_n(R)\phi_n(\alpha)$

Eigenfunctions of $-\partial^2/\partial\alpha^2$:

$$-\frac{\partial^2}{\partial\alpha^2}\phi_n(\alpha)=s_n^2\phi_n(\alpha) \quad \Rightarrow \quad \phi_n(\alpha)=\sin\left(s_n\left(\frac{\pi}{2}-\alpha\right)\right)$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}+\frac{s_n^2}{R^2}-k^2\right)F_n(R)=0$$

$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

Derivation for three identical bosons

Effective Schrödinger equation for the hyper-radius R

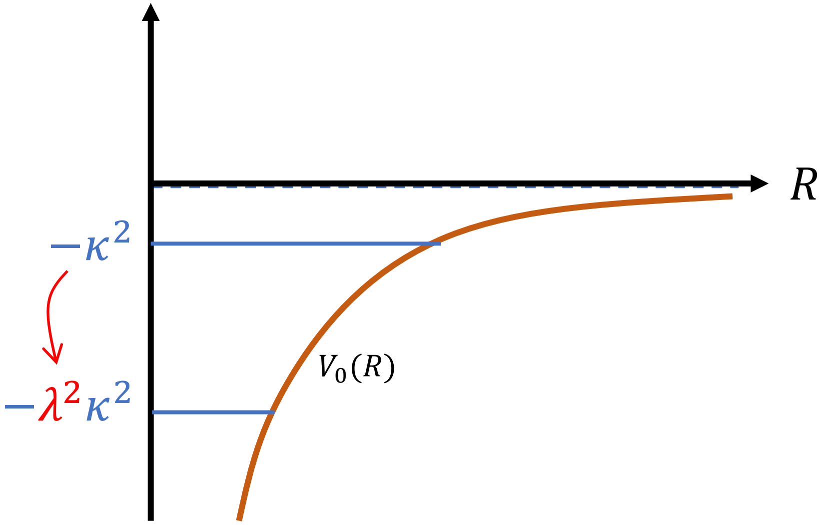
$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

All s_n are real, except one: $s_0=\pm i1.00624$

For $n=0$, one gets the Efimov attractive potential $\boxed{V_0(R)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}$

Scale invariance
$R\rightarrow\lambda R$



Solution F at energy $-\kappa^2$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\kappa^2\right)\sqrt{R}F_n(R)=0$$

Solution F at energy $-\lambda^2\kappa^2$

$$\left(-\frac{\partial^2}{\partial R^2}+\frac{s_n^2-\frac{1}{4}}{R^2}+\lambda^2\kappa^2\right)\sqrt{\lambda R}F_n(\lambda R)=0$$

Without boundary condition at short hyper-radius, the Efimov attraction allows any negative energy : **this illustrates the Thomas collapse!**

Derivation for three identical bosons

Effective Schrödinger equation for the hyper-radius R

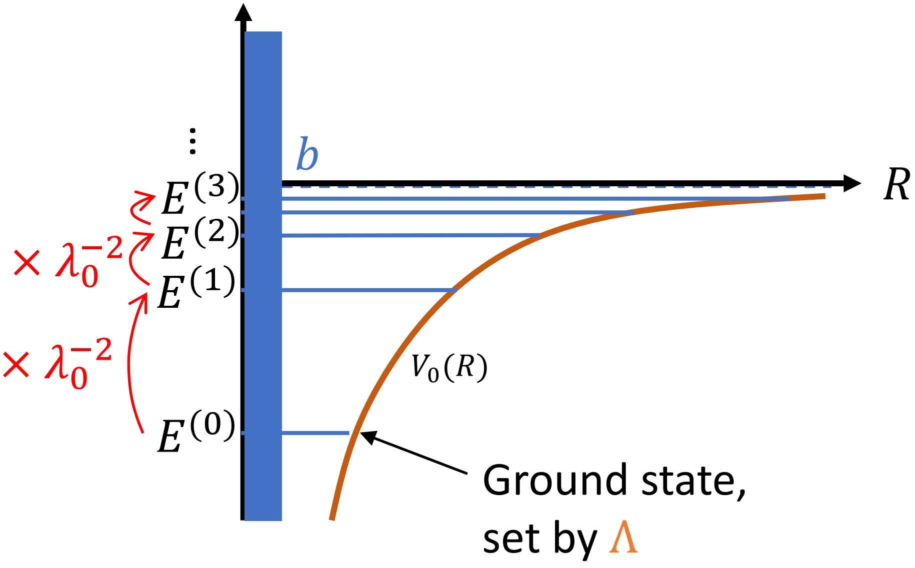
$$\left(-\frac{\partial^2}{\partial R^2}+\underbrace{\boxed{\frac{s_n^2-\frac{1}{4}}{R^2}}}_{V_n(R)}-k^2\right)\sqrt{R}F_n(R)=0$$

$$s_n\cos\left(\frac{s_n\pi}{2}\right)+\frac{8}{\sqrt{3}}\sin\left(\frac{s_n\pi}{6}\right)=0$$

All s_n are real, except one: $s_0 = \pm i1.00624$

For $n = 0$, one gets the Efimov attractive potential $\boxed{V_0(R) = -\frac{|s_0|^2+\frac{1}{4}}{R^2}}$

Discrete scale invariance
$R \to \lambda_0 R$



Ground state, set by Λ

The problem is due to the zero-range approximation. In reality, when the three particles come within distances of the order of b, the interaction potential sets a boundary condition for R.

For small R, $F_0(R) = \alpha R^{i|s_0|} + \beta R^{-i|s_0|} \propto \cos(|s_0|\ln \Lambda R)$

Three-body parameter

$$F_0(\lambda R) \propto \cos(|s_0|\ln \Lambda\lambda R) = \cos(|s_0|\ln \Lambda R + \underbrace{|s_0|\ln \lambda}_{\pi}) \quad \propto F_0(R)$$

$$\boxed{\lambda_0 = e^{\pi/|s_0|} \approx 22.7} \Rightarrow \boxed{E^{(n)} = E^{(0)}\lambda_0^{-2n}}$$

Generalised discrete scaling away from unitarity

Suppose we have a solution χ_0 of:

Equation:

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}+\kappa^2\right)\chi_0(R,\alpha)=0$$

Boundary condition $\alpha \to 0$:

$$\left[\frac{\partial}{\partial\alpha}\chi_0(R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(R,\frac{\pi}{3}\right)=-\frac{R}{a}\chi_0(R,0)$$

Discrete scaling: $R \to R/\lambda_0$

$$\left(-\frac{\partial^2}{\partial R^2}-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial \alpha^2}+\boxed{\lambda_0^{-2}\kappa^2}\right)\chi_0(\lambda_0 R,\alpha)=0$$

$$\left[\frac{\partial}{\partial\alpha}\chi_0(\lambda_0 R,\alpha)\right]_{\alpha\to 0}+\frac{8}{\sqrt{3}}\chi_0\left(\lambda_0 R,\frac{\pi}{3}\right)=-\frac{R}{\boxed{\lambda_0 a}}\chi_0(\lambda_0 R,0)$$

i.e. we have a new solution for:

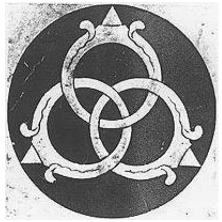
$a \to \lambda_0 a$
$\kappa \to \kappa/\lambda_0$

$$(a^{-1},\kappa)\to(a^{-1},\kappa)/\lambda_0$$

“Efimov spectrum”

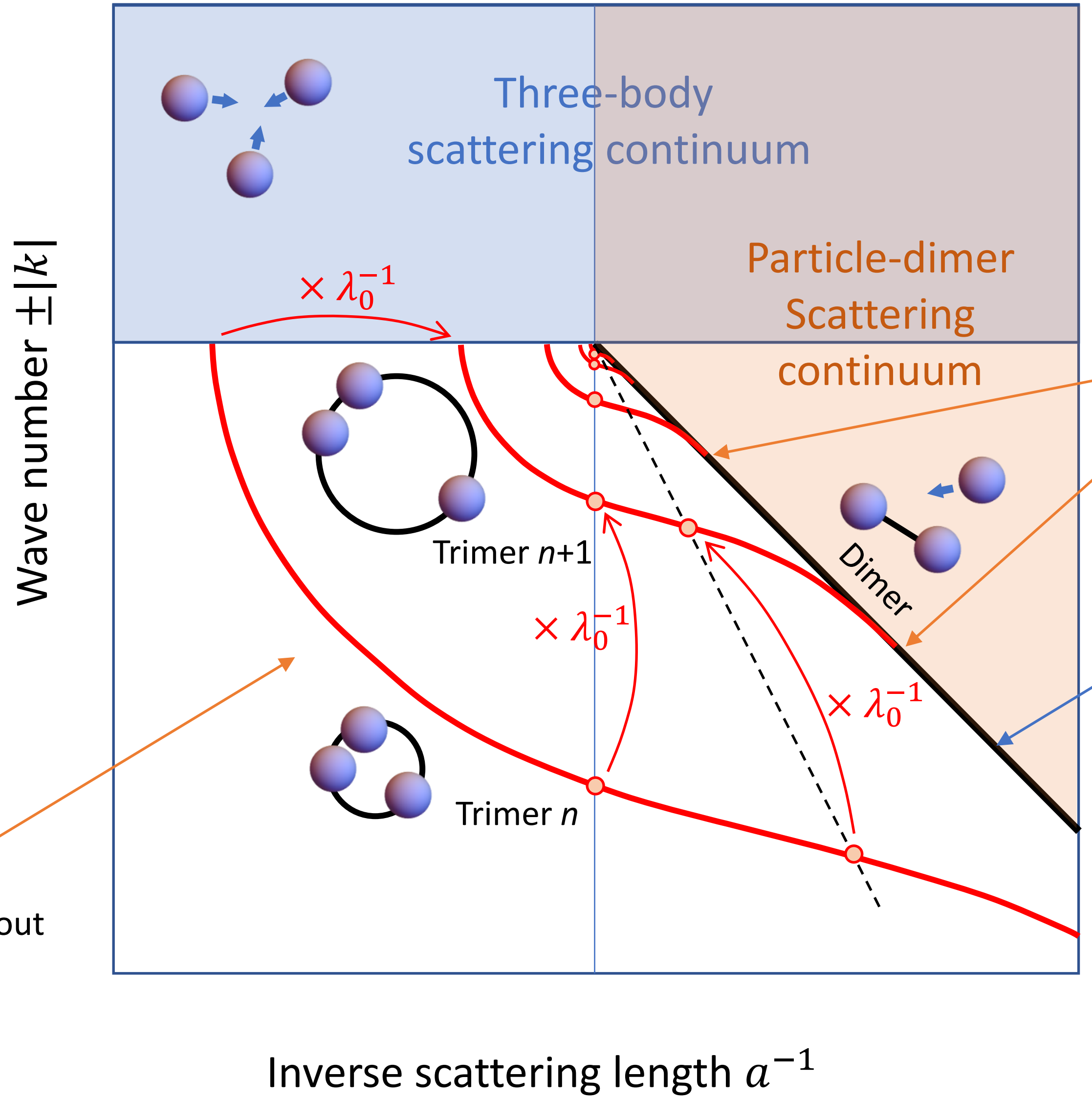
1. Discrete scale invariance

Infinite number of three-body bound states.



2. Borromean states

Three-body bound states without two-body bound states.

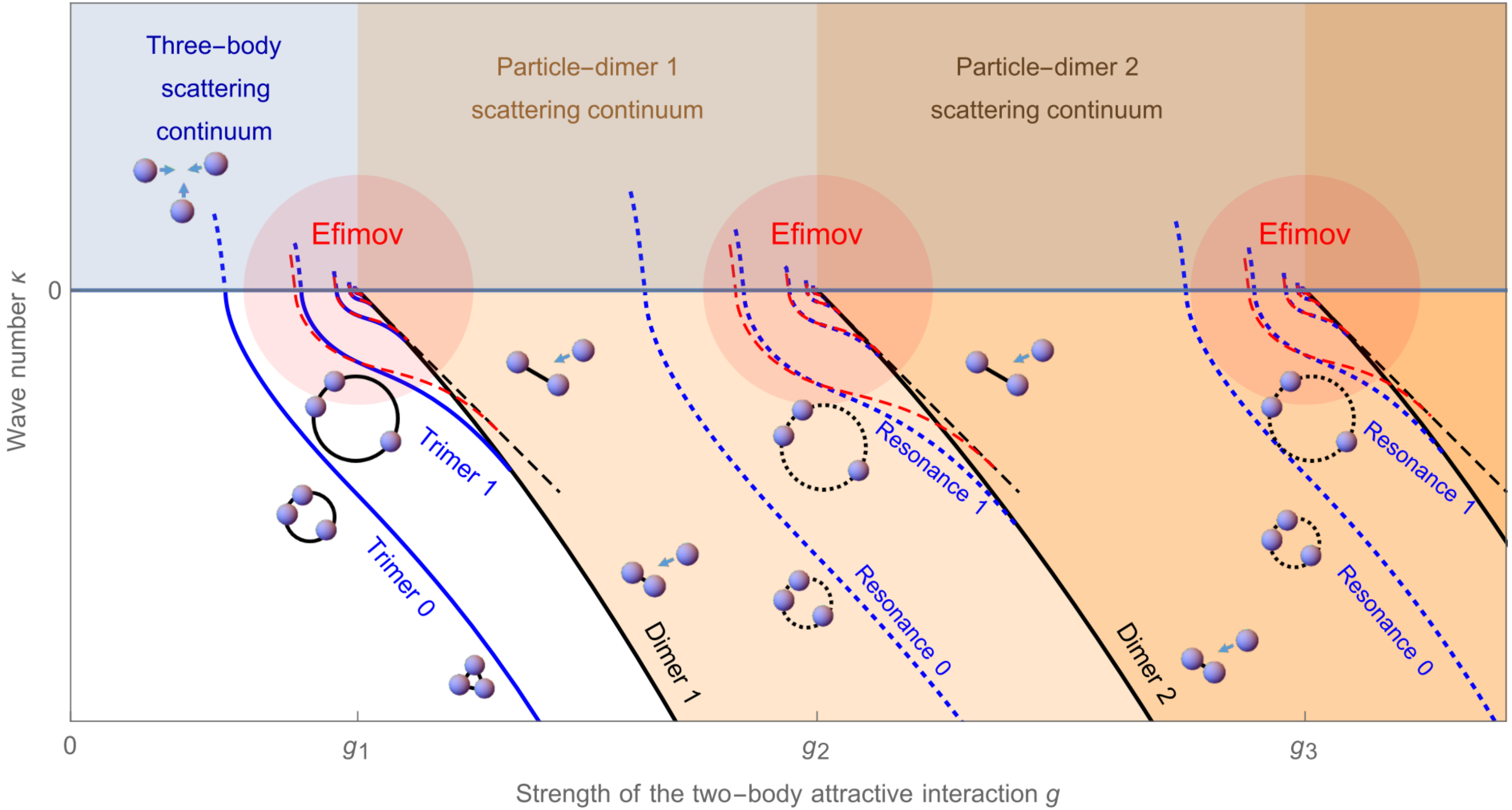


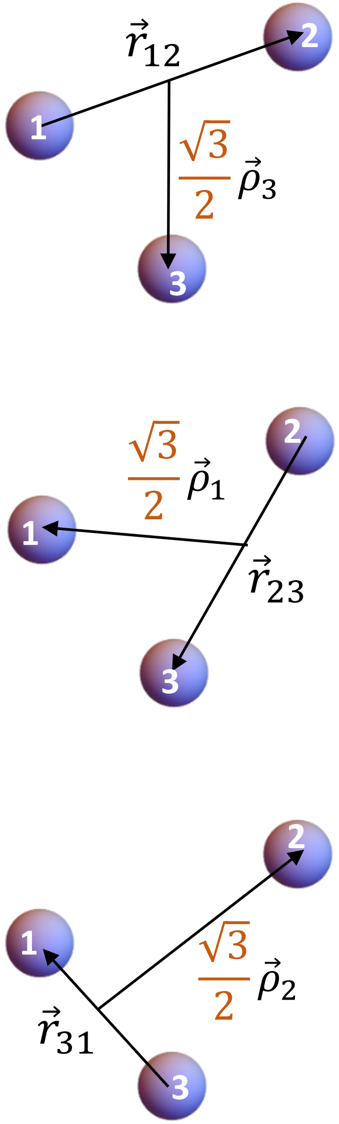
3. Trimer dissociation with increasing interaction

$$E_2 = -\frac{\hbar^2\kappa^2}{m}$$

with $\kappa = a^{-1}$

Three bosons





What is the shape of an Efimov state?

No definite shape (fluctuating) but a tendency to form elongated triangles

$$\begin{aligned}\Psi &= \chi(\vec{r}_{12},\vec{\rho}_3)+\chi(\vec{r}_{23},\vec{\rho}_1)+\chi(\vec{r}_{31},\vec{\rho}_2)\\ &= \frac{\chi_0(r_{12},\rho_3)}{r_{12}\;\rho_3}+\frac{\chi_0(r_{23},\rho_1)}{r_{23}\;\rho_1}+\frac{\chi_0(r_{31},\rho_2)}{r_{31}\;\rho_2}\\ &= \frac{2}{R^2}\left(\frac{\chi_0(R,\alpha_3)}{\sin 2\alpha_3}+\frac{\chi_0(R,\alpha_1)}{\sin 2\alpha_1}+\frac{\chi_0(R,\alpha_2)}{\sin 2\alpha_2}\right)\\ &= \underbrace{\frac{2F(R)}{R^2}}_{\substack{\text{Hyper-radial}\\ \text{(size)}}}\underbrace{\left(\frac{\phi_0(\alpha_3)}{\sin 2\alpha_3}+\frac{\phi_0(\alpha_1)}{\sin 2\alpha_1}+\frac{\phi_0(\alpha_2)}{\sin 2\alpha_2}\right)}_{\substack{\text{Hyper-angular}\\ \text{(shape)}}}\end{aligned}$$

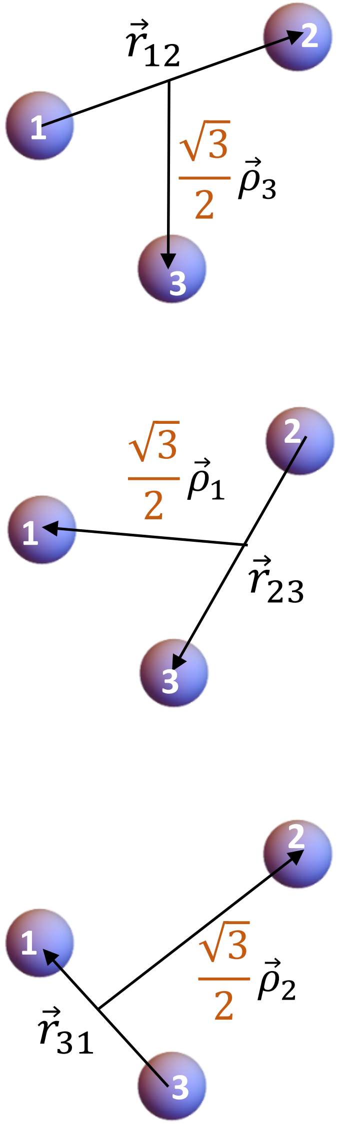
Partial wave $L = 0$

Hyperspherical coordinates:

$$r = R\sin\alpha$$
$$\rho = R\cos\alpha$$

$\chi_0(R,\alpha) = F(R)\phi_0(\alpha)$ at unitarity

with $\phi_0(\alpha) = \sinh\left(|s_0|\left(\frac{\pi}{2}-\alpha\right)\right)$



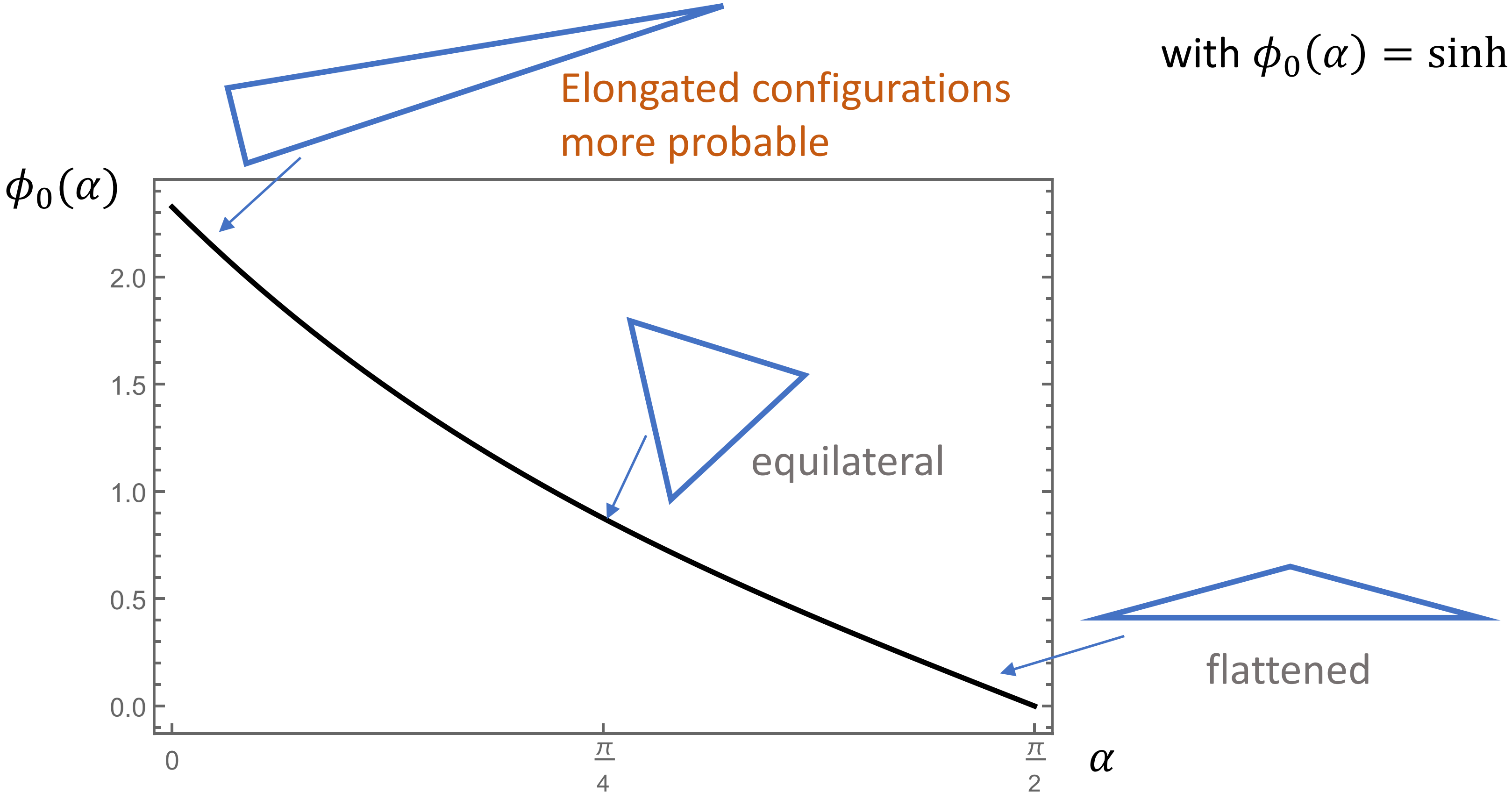
What is the shape of an Efimov state?

No definite shape (fluctuating) but a tendency to form elongated triangles

$$\Psi = \frac{2F(R)}{R^2}\left(\frac{\phi_0(\alpha_3)}{\sin 2\alpha_3} + \frac{\phi_0(\alpha_1)}{\sin 2\alpha_1} + \frac{\phi_0(\alpha_2)}{\sin 2\alpha_2}\right)$$

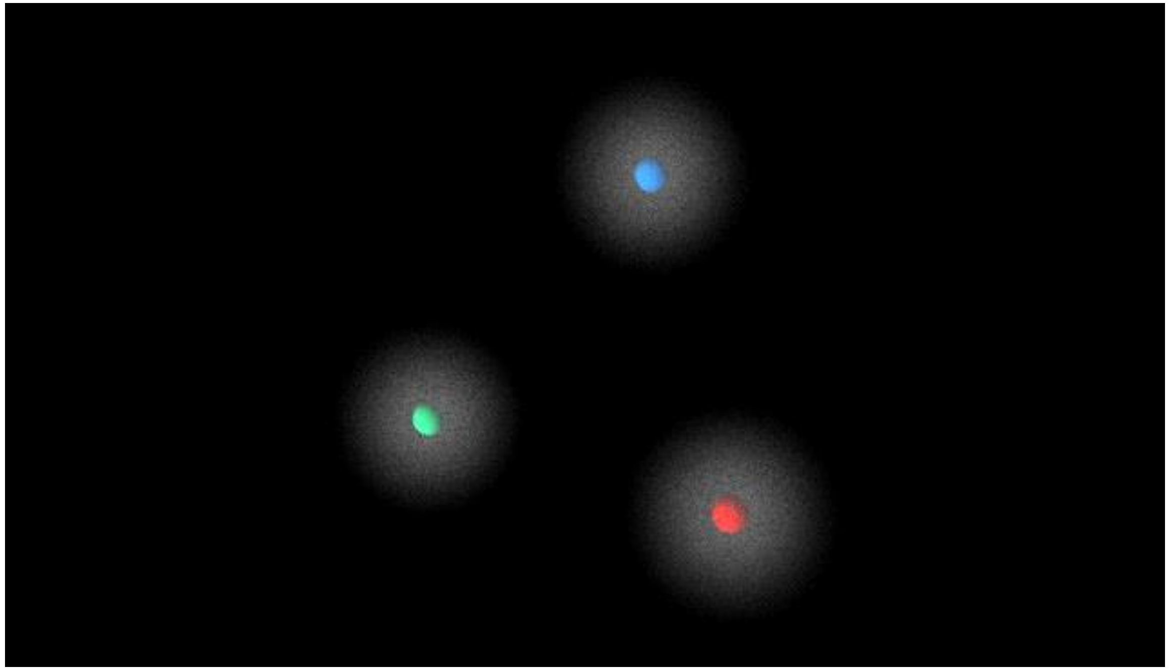
$$r = R \sin \alpha$$
$$\rho = R \cos \alpha$$

with $\phi_0(\alpha) = \sinh\left(|s_0|\left(\frac{\pi}{2} - \alpha\right)\right)$


Elongated configurations more probable

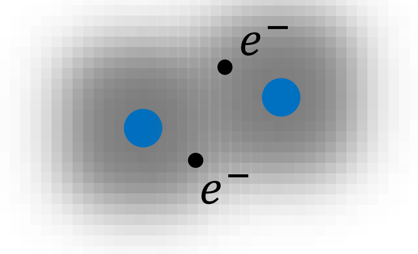
Why long range ? in spite of short-range two-body interactions?

The Efimov attraction may be viewed as an interaction between two particles mediated by a third particle

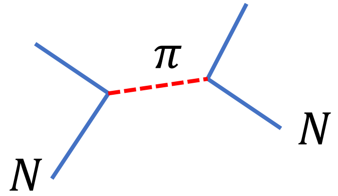


Similar to:

Chemical covalent bonding
(exchange of electron)



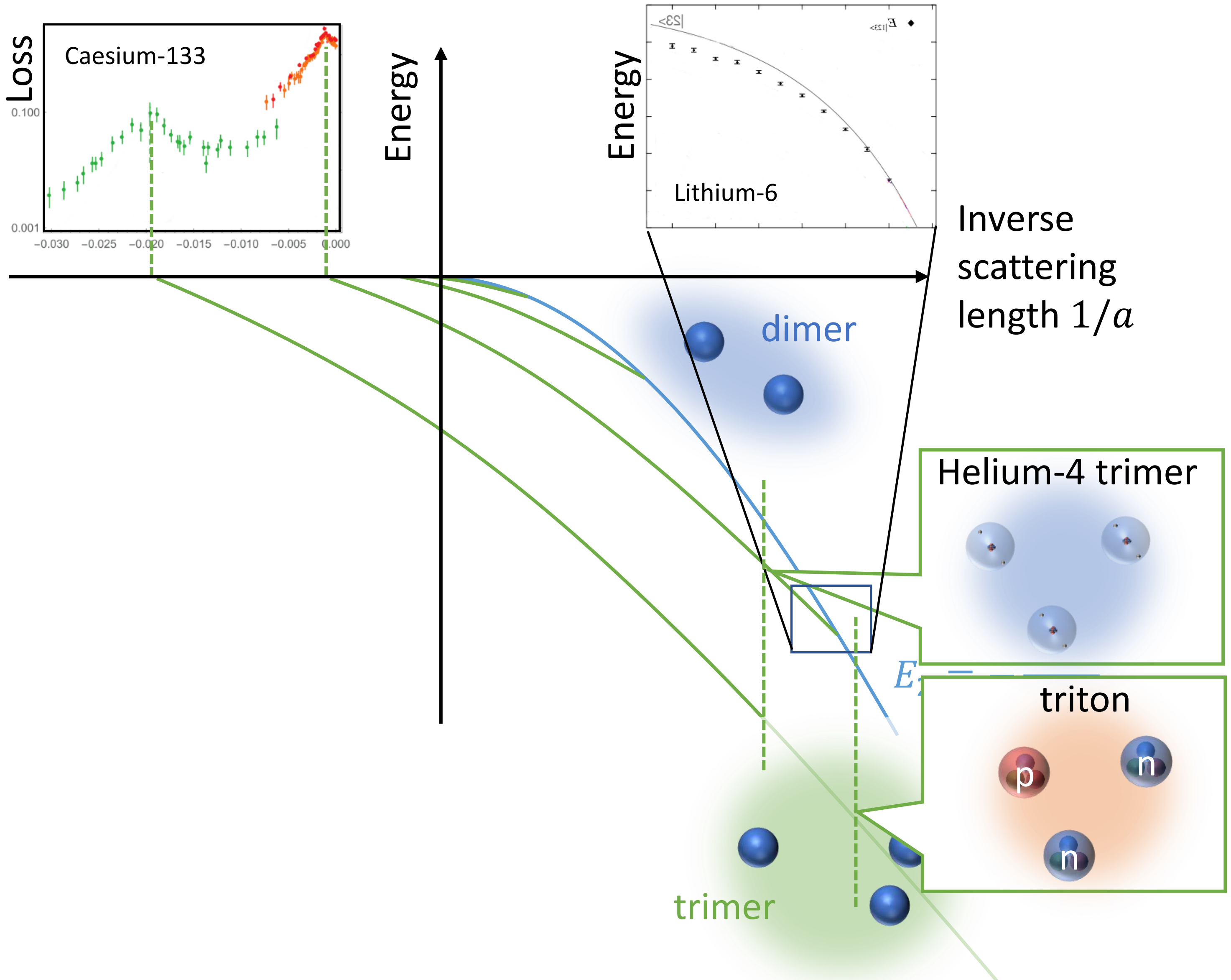
Nuclear force
(exchange of virtual meson)



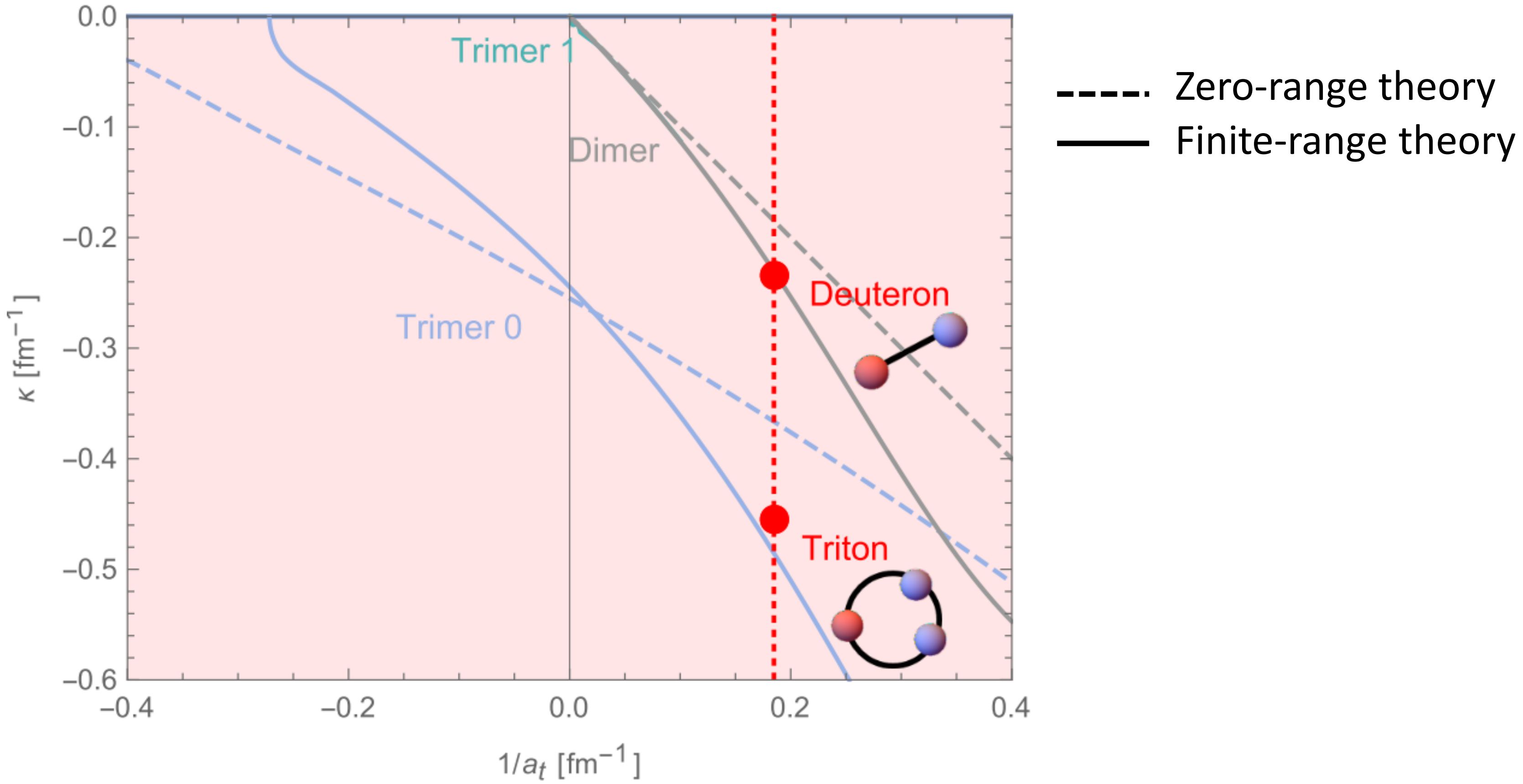
Overview of universal clusters

Theoretical predictions
Experimental observations

Three bosons



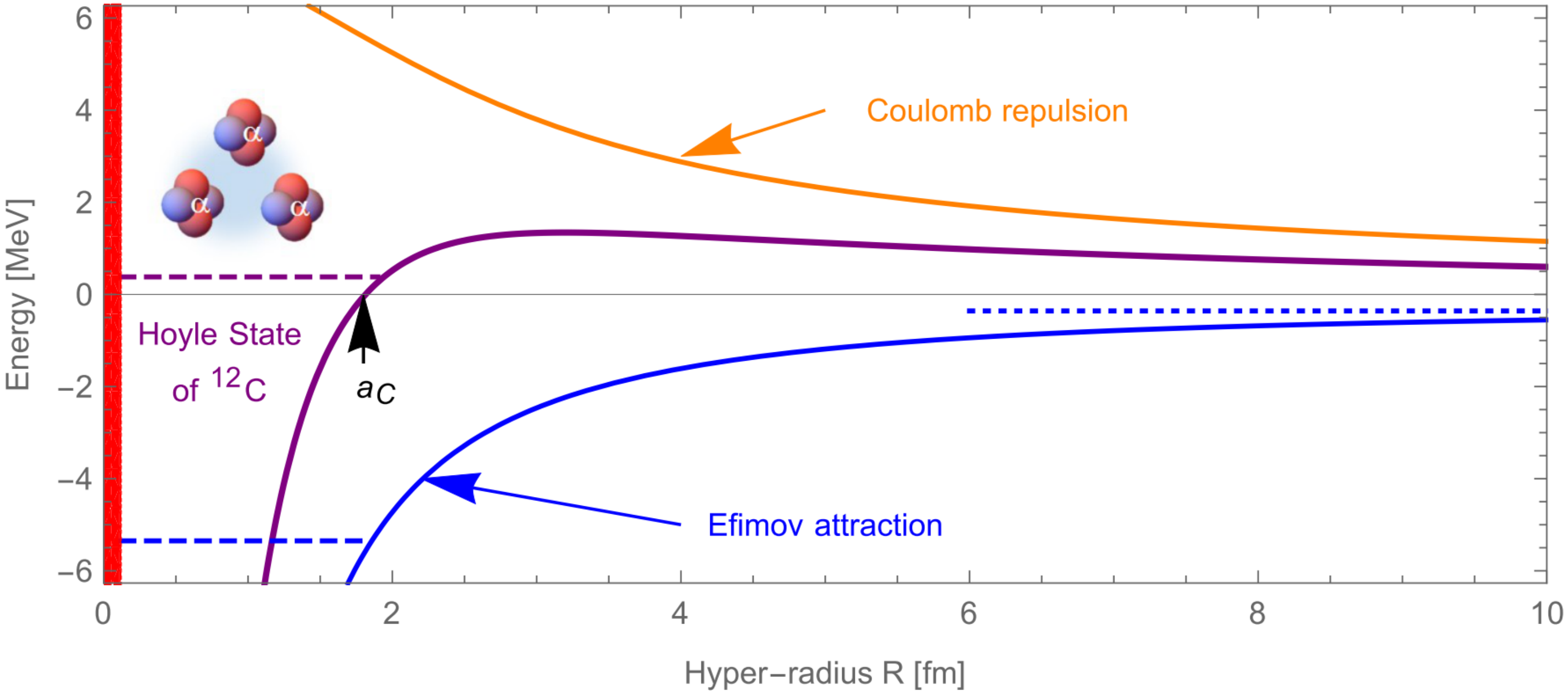
The triton (2 neutrons + 1 proton)



Qualitatively consistent,
but not the best example.

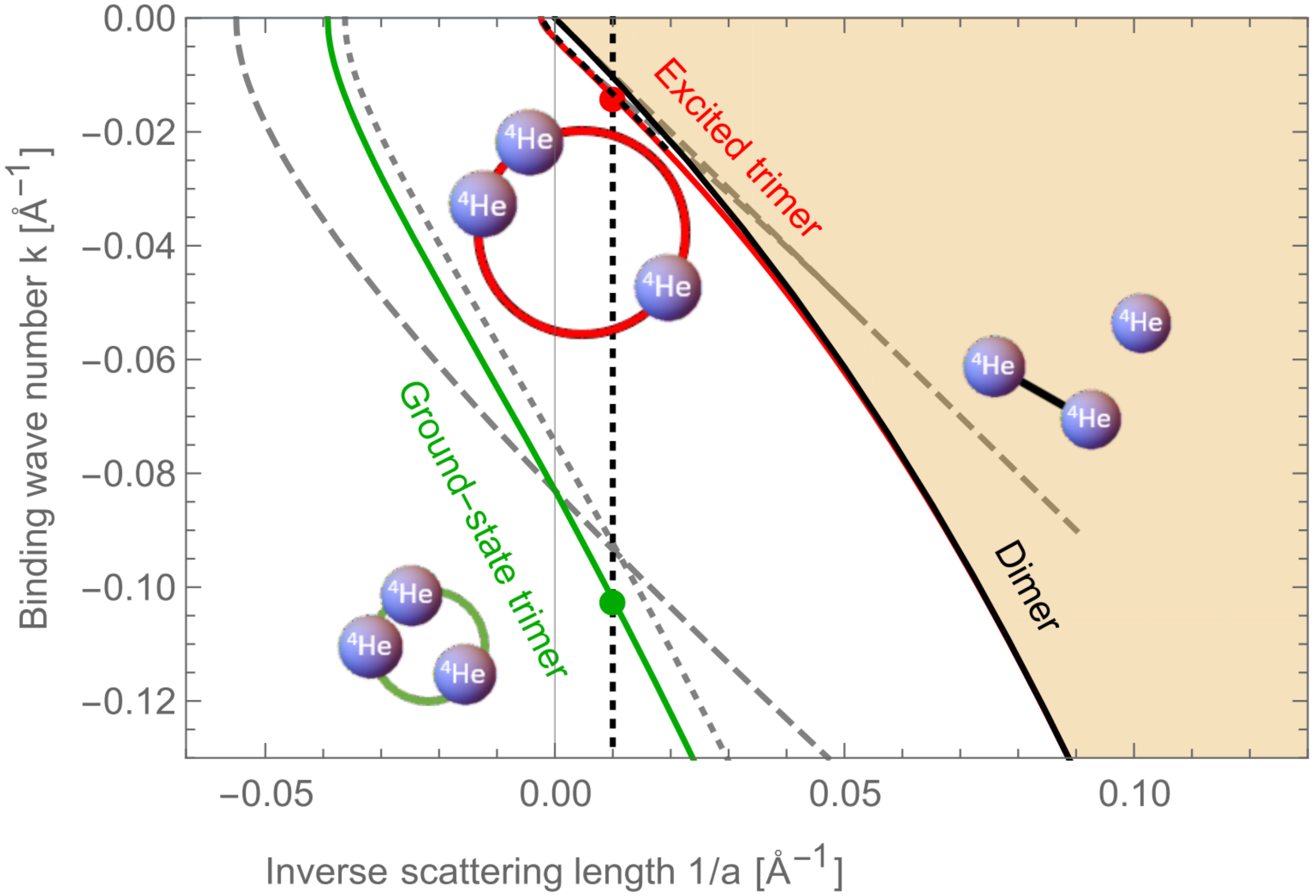
The Hoyle state of carbon-12

Originally suggested in V. Efimov's first paper



No clear evidence.

The helium triatomic molecules $^4He_3$



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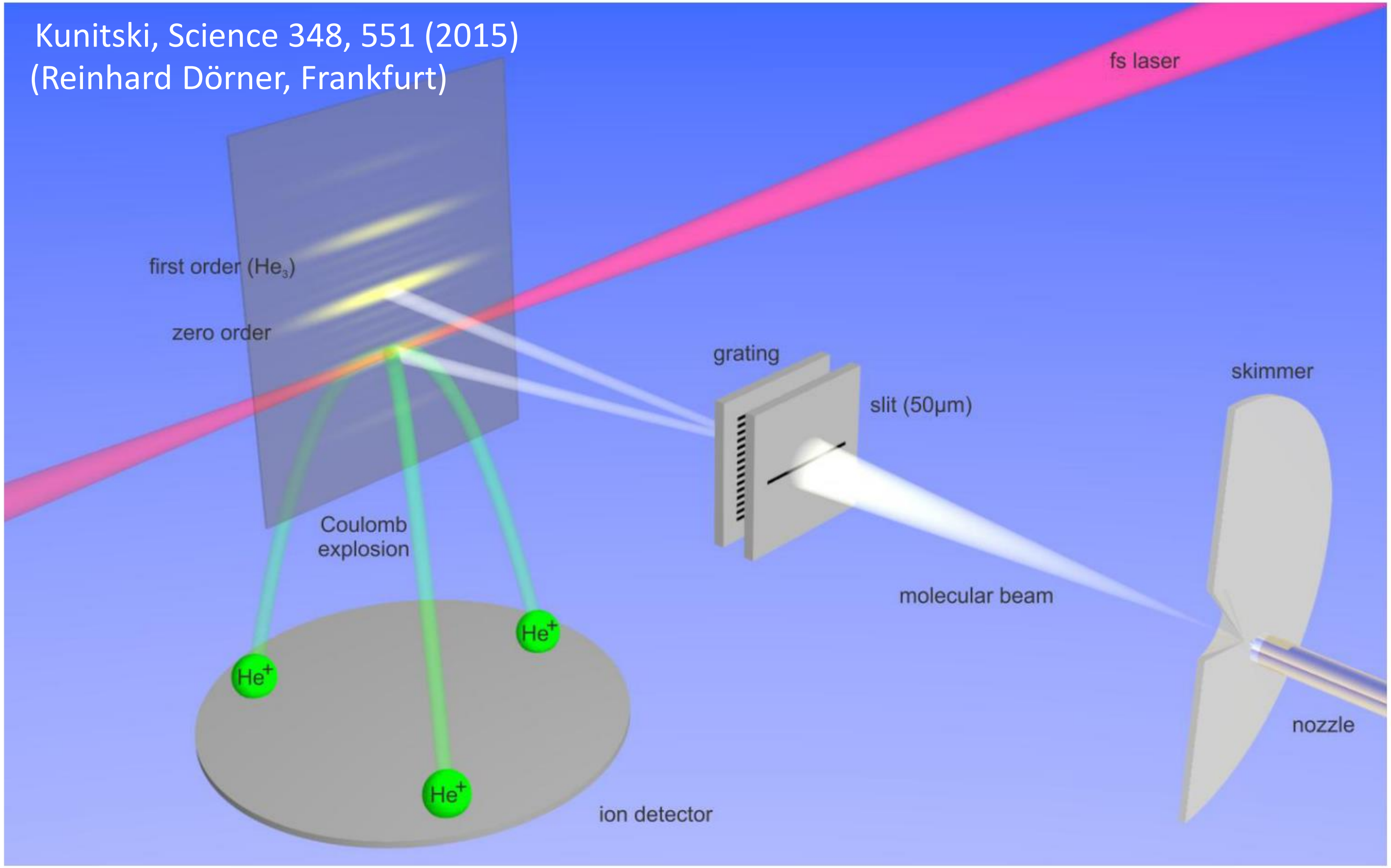

Kunitski, Science 348, 551 (2015)
(Reinhard Dörner, Frankfurt)

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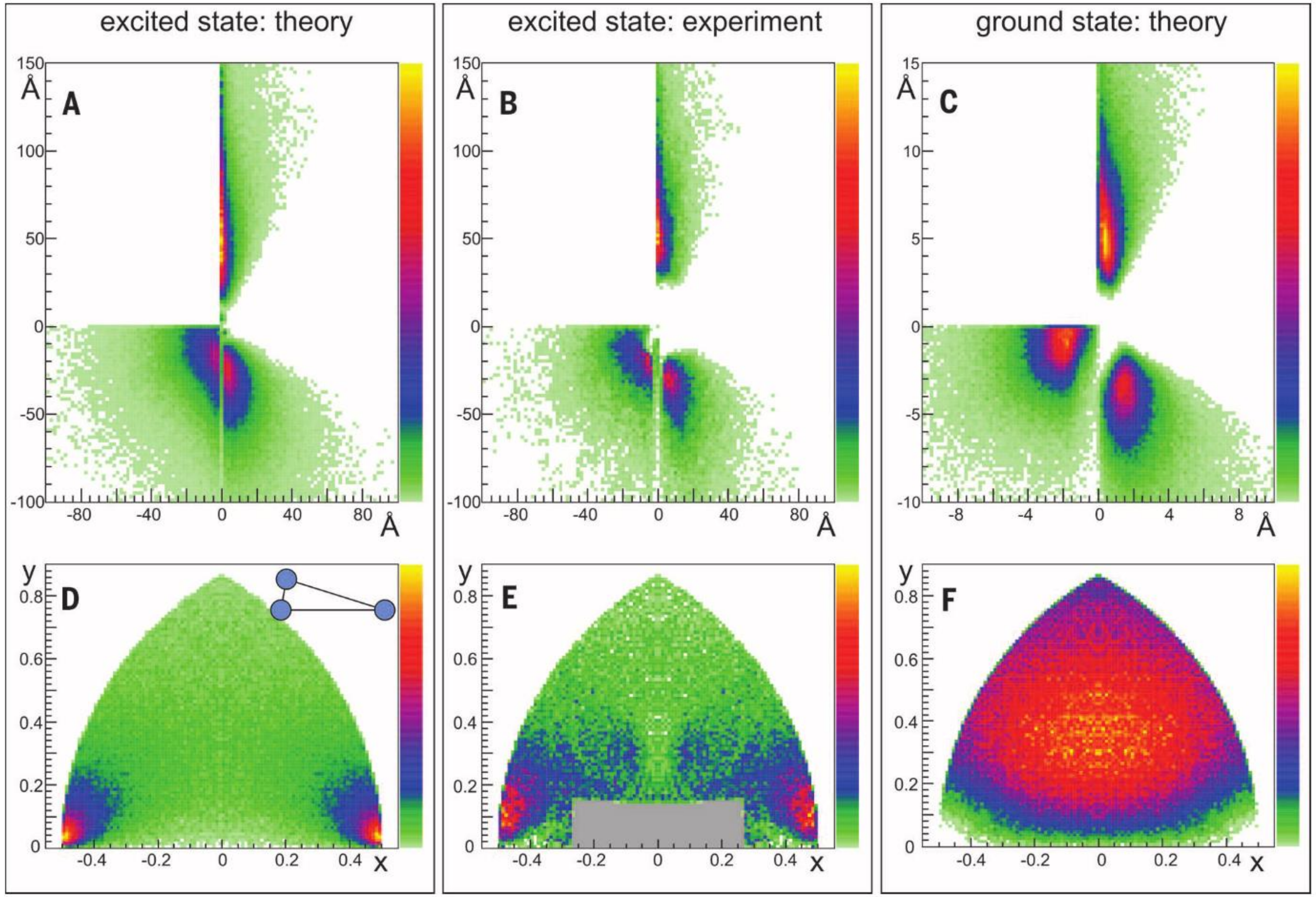
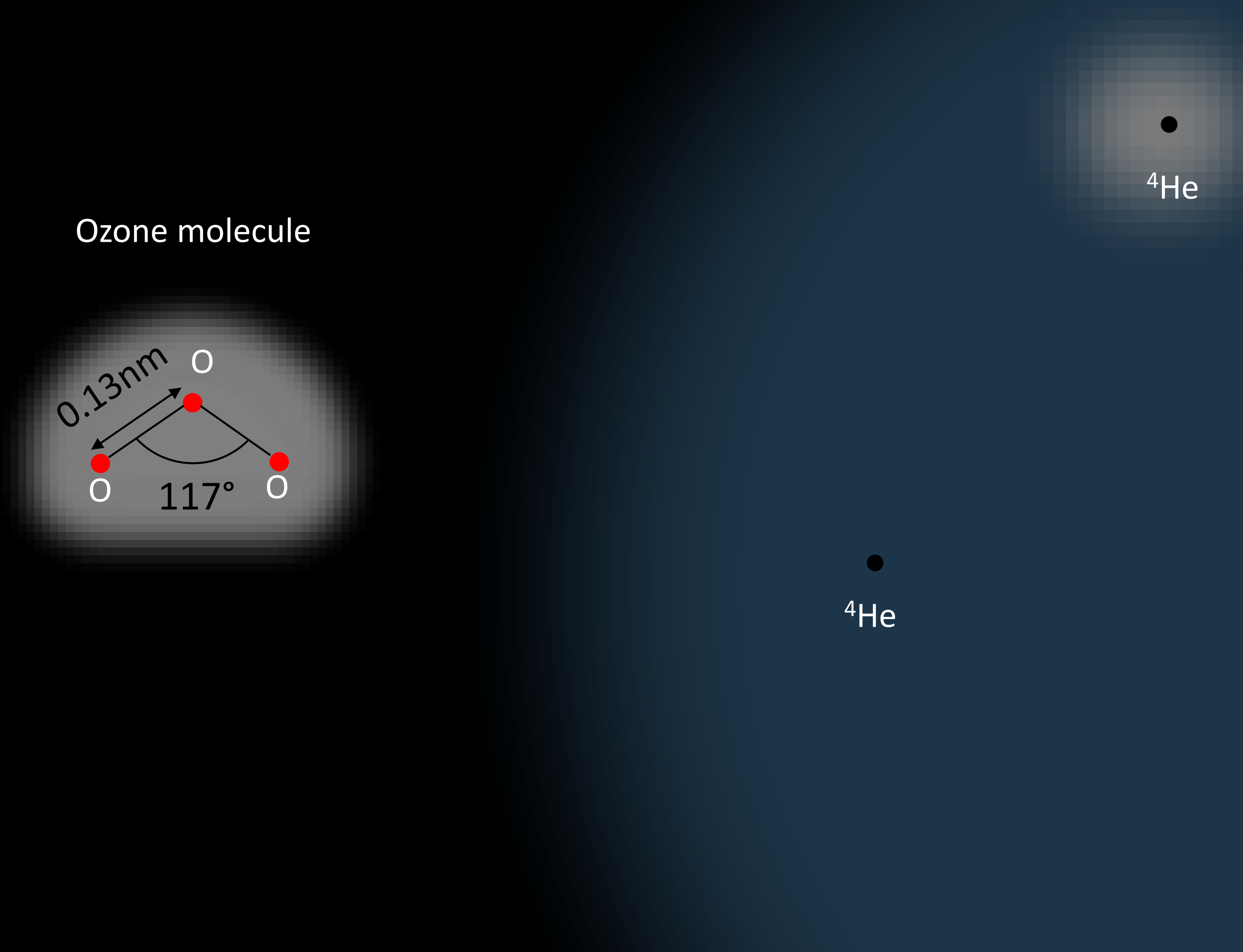
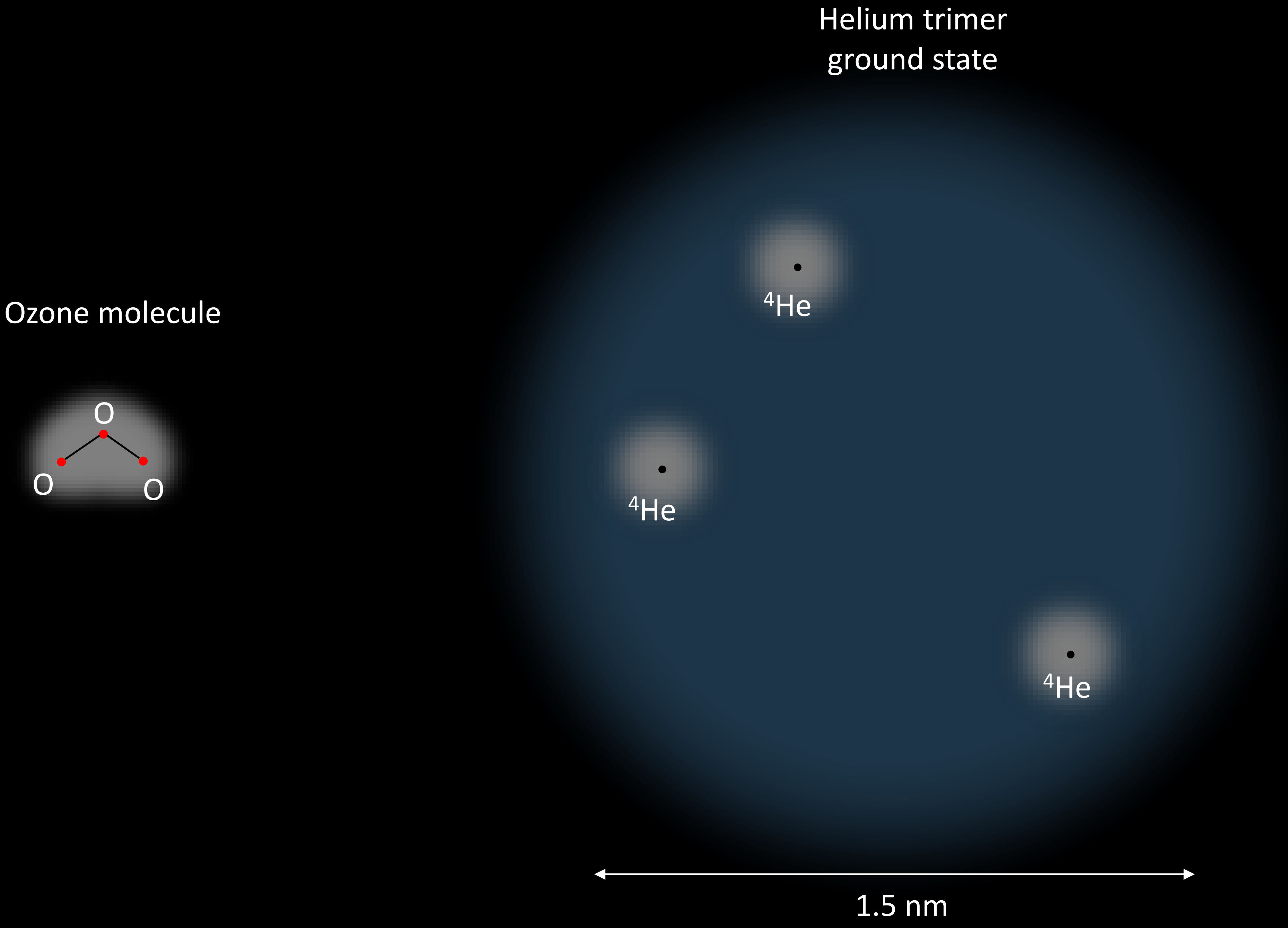


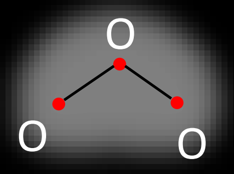
Ozone molecule

Helium trimer ground state

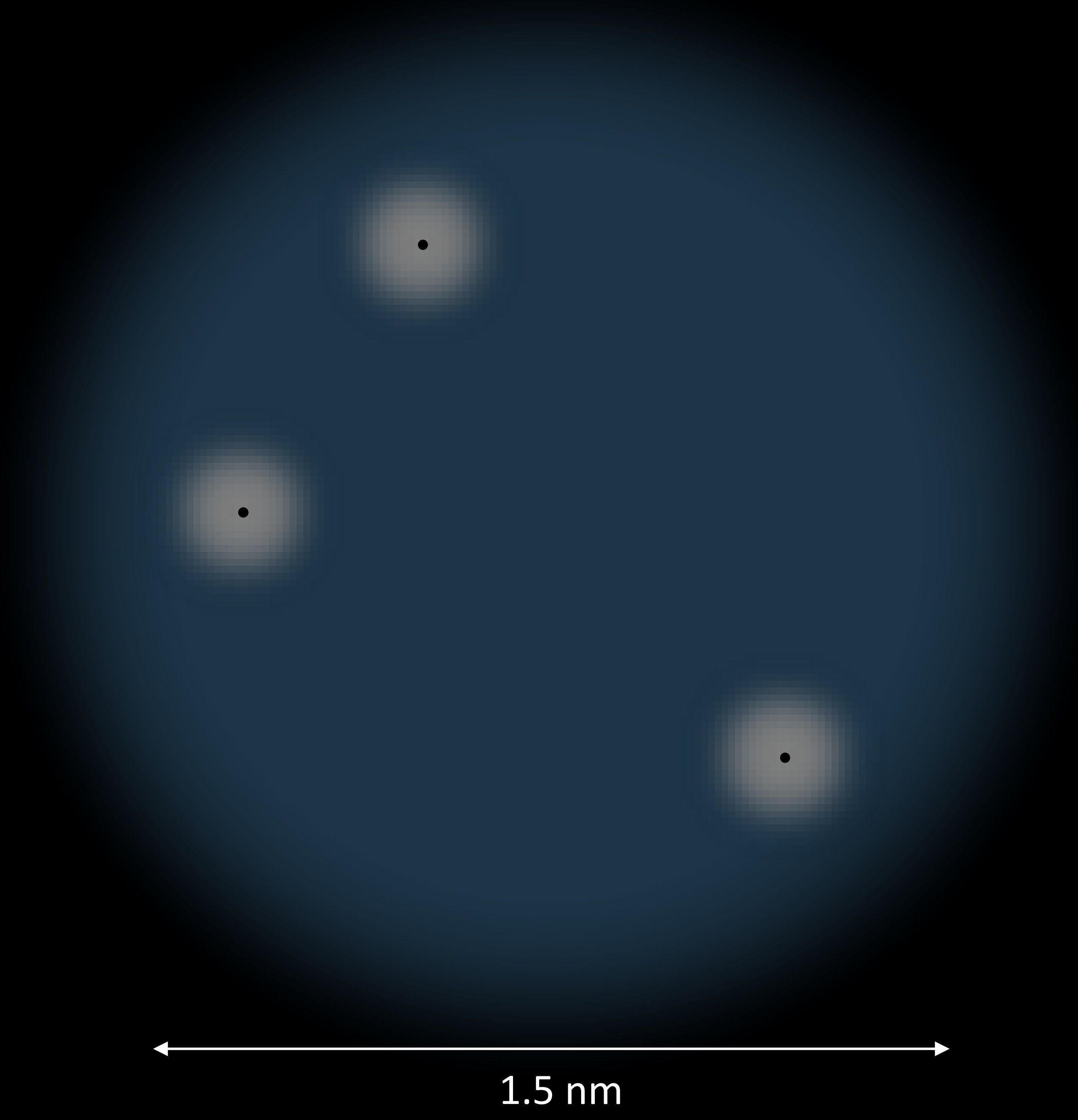



Ozone molecule
O
O
O
Helium trimer
ground state
^{4}He
^{4}He
^{4}He
1.5 nm

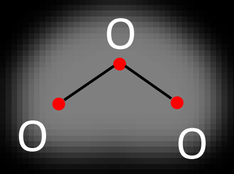
Ozone molecule



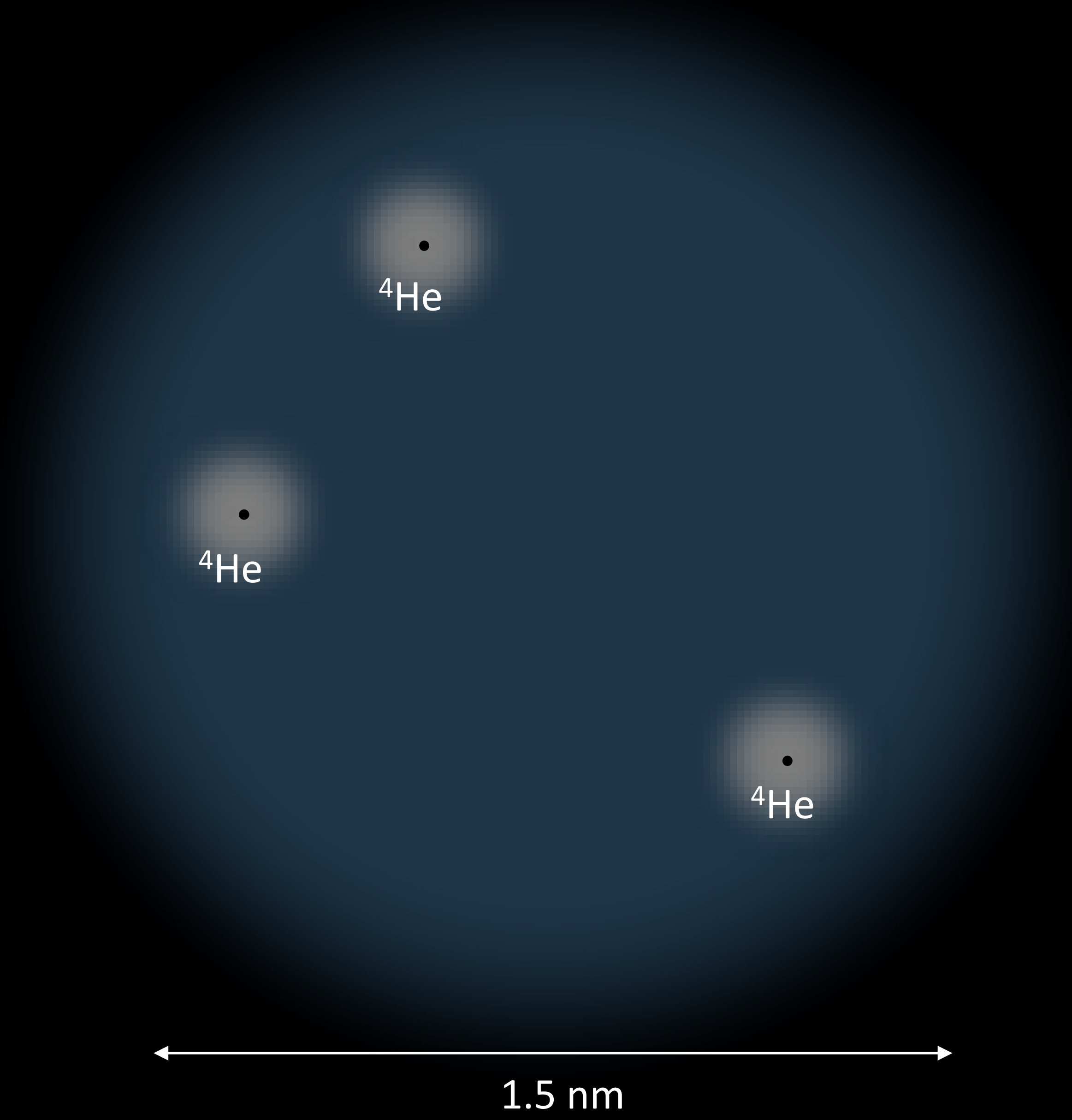
Helium trimer ground state

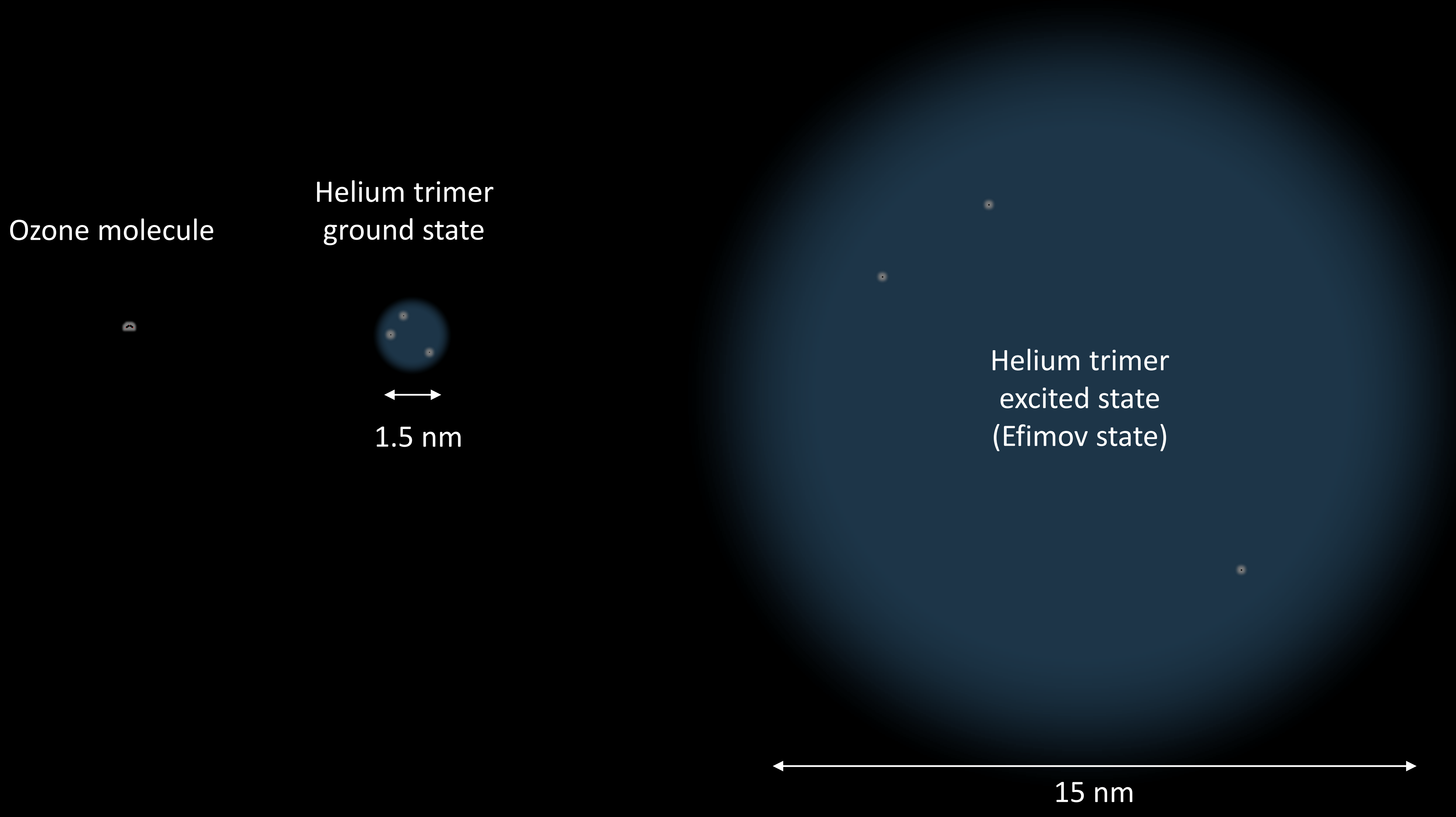


Ozone molecule



Helium trimer
ground state





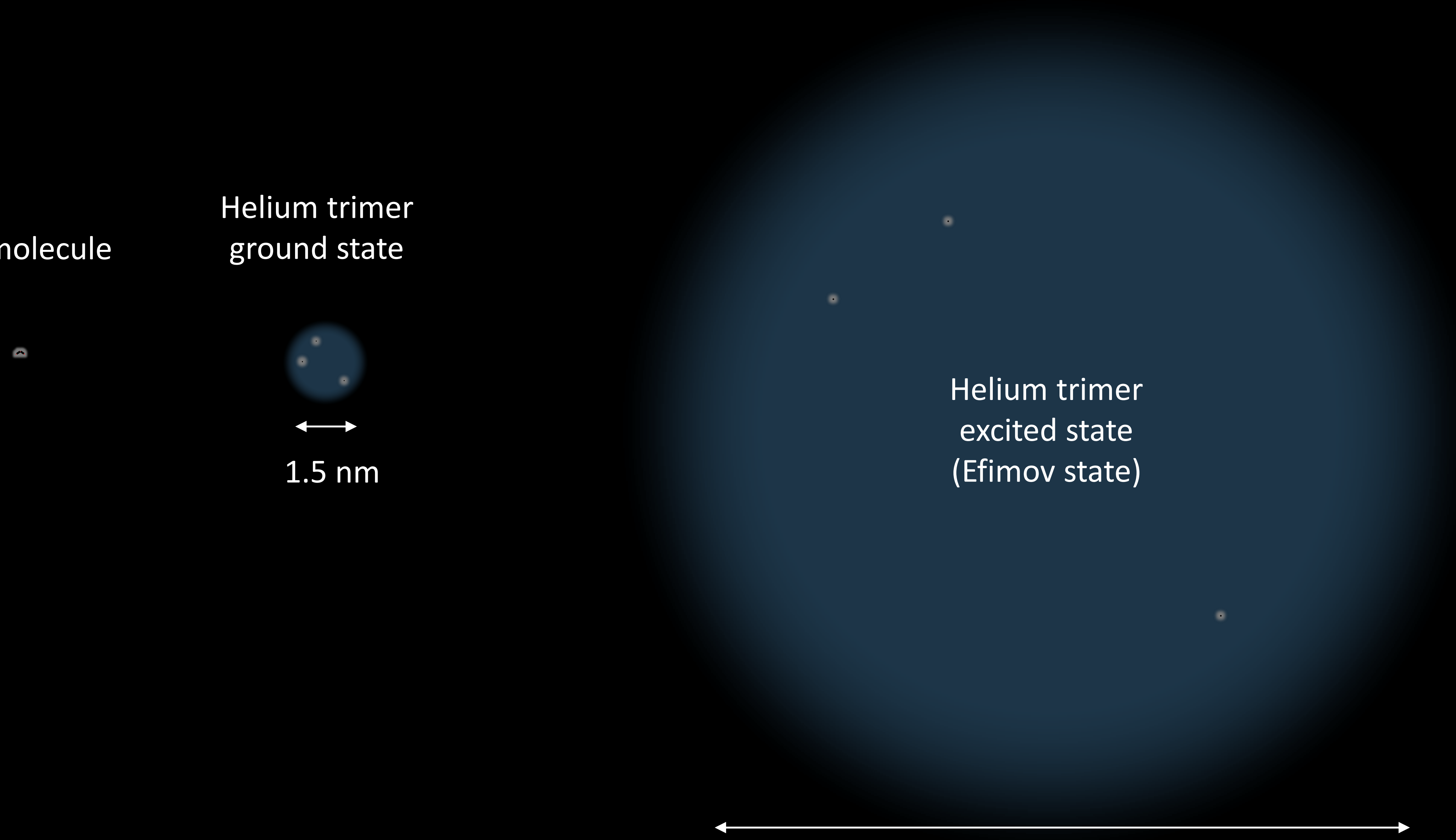
Ozone molecule

Helium trimer
ground state

1.5 nm

Helium trimer
excited state
(Efimov state)

15 nm



Ozone molecule

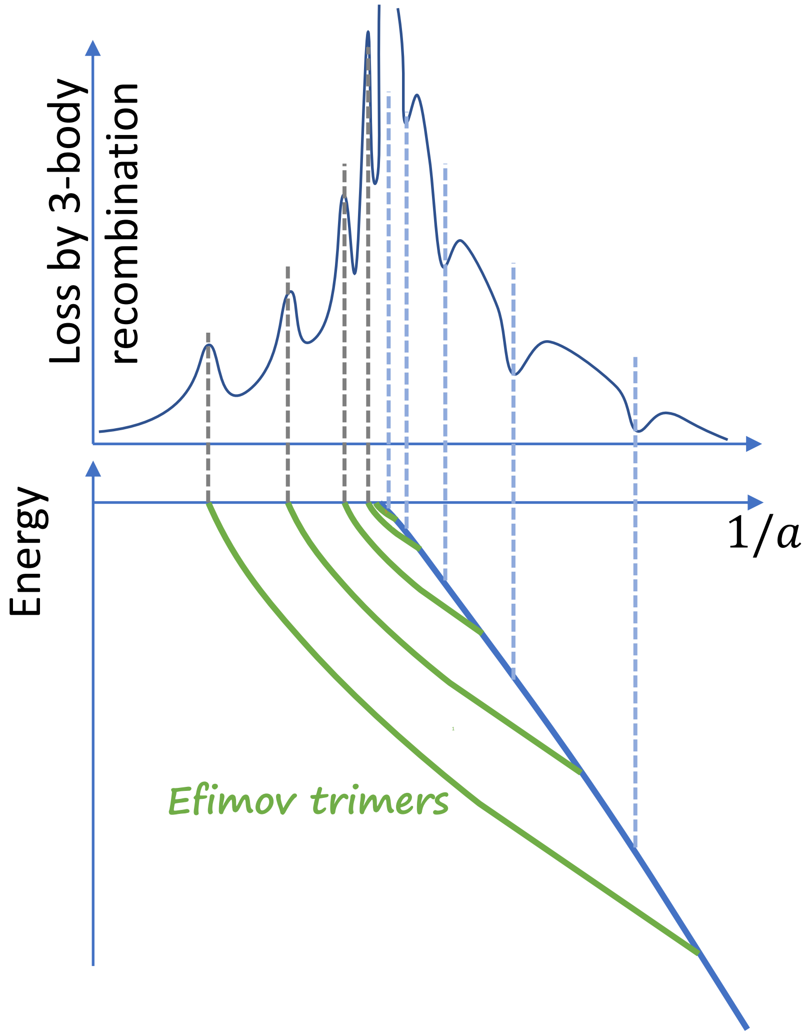
Helium trimer
ground state

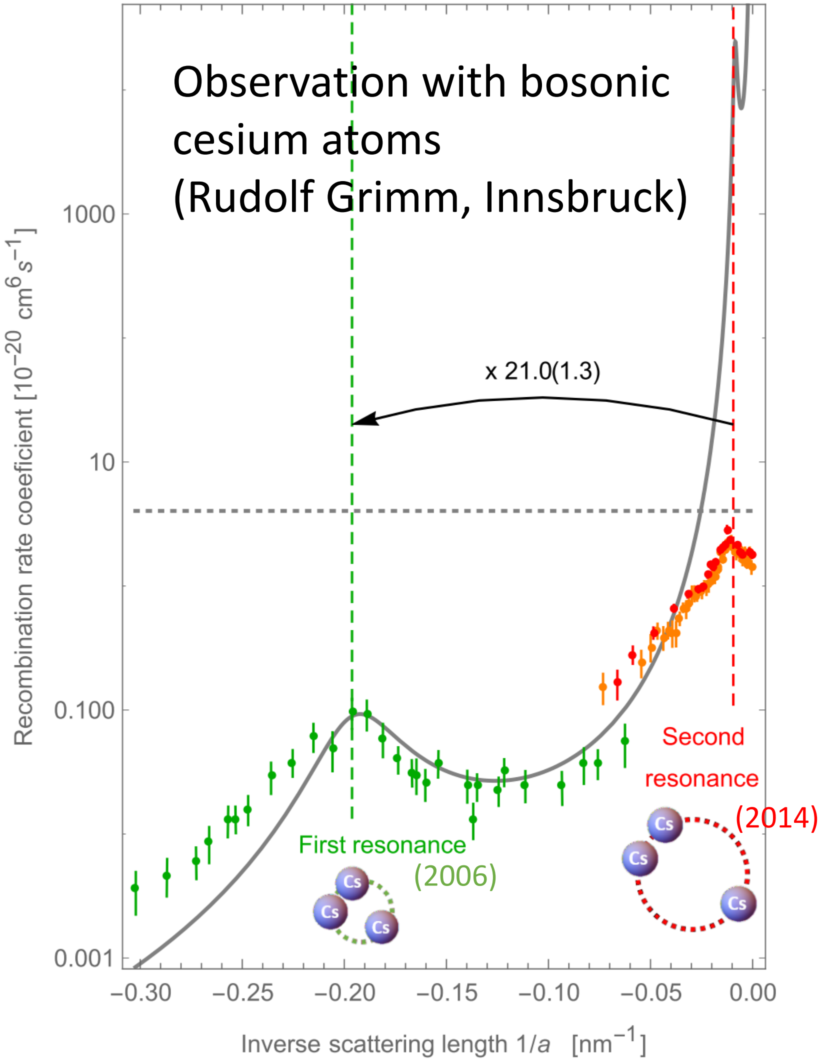
1.5 nm

Helium trimer
excited state
(Efimov state)

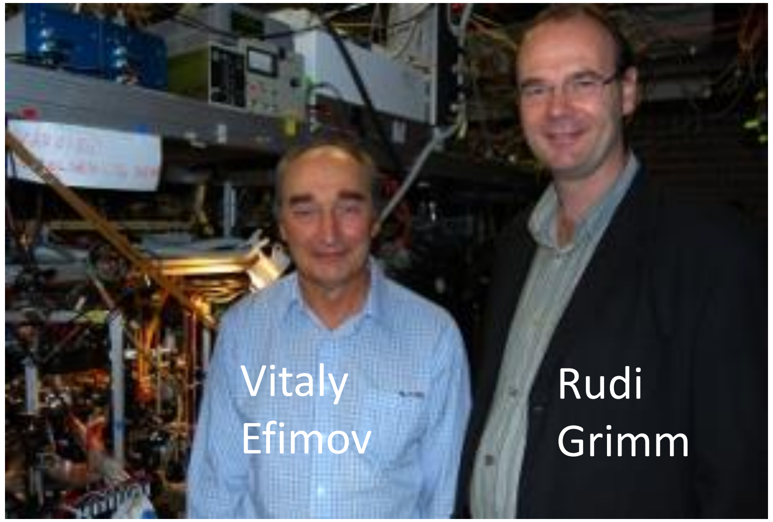
15 nm

Observations in ultra-cold atomic gases





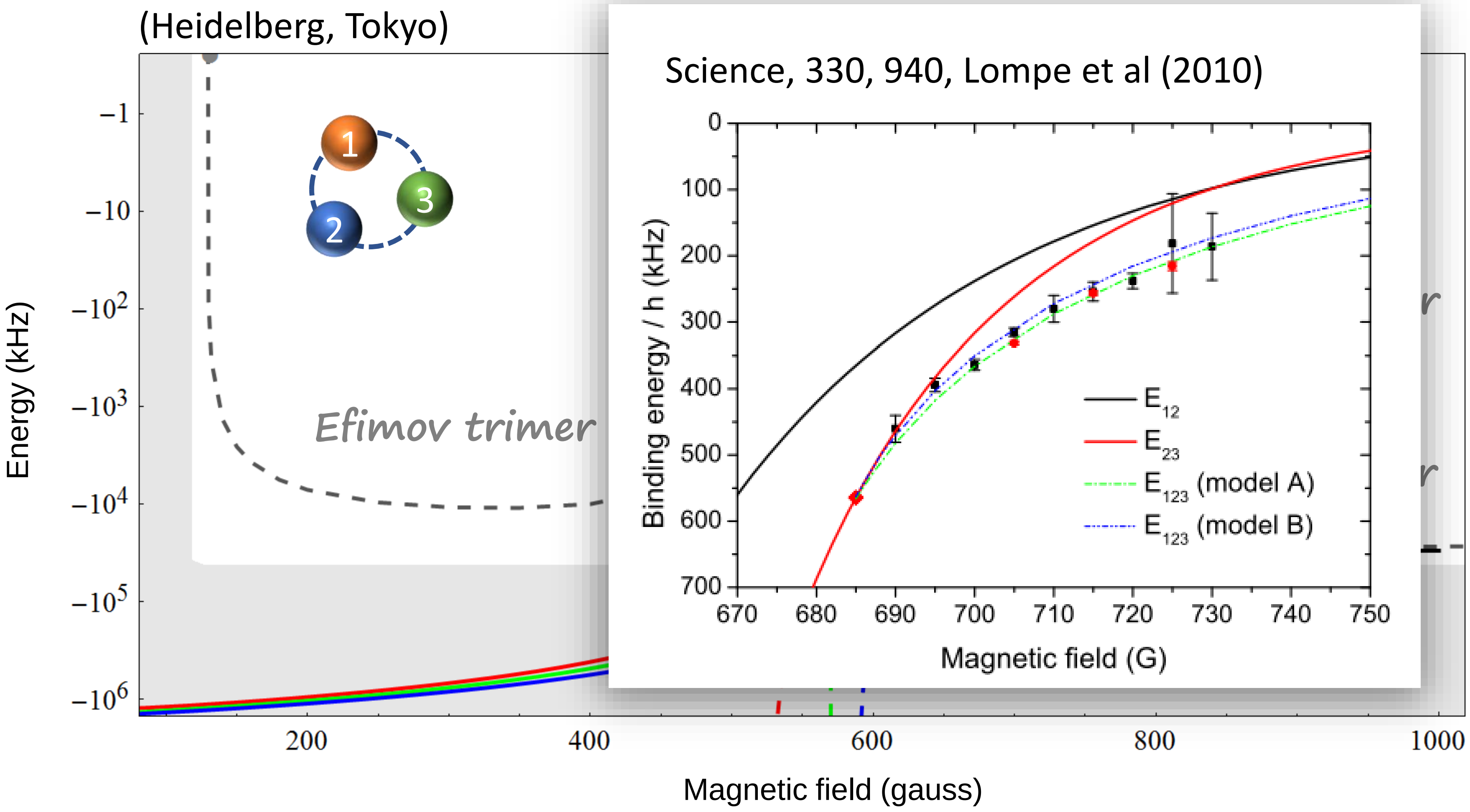
Observation with bosonic cesium atoms (Rudolf Grimm, Innsbruck)



Vitaly Efimov and Rudolf Grimm receive the first Faddeev medal in Caen (July 11, 2018)

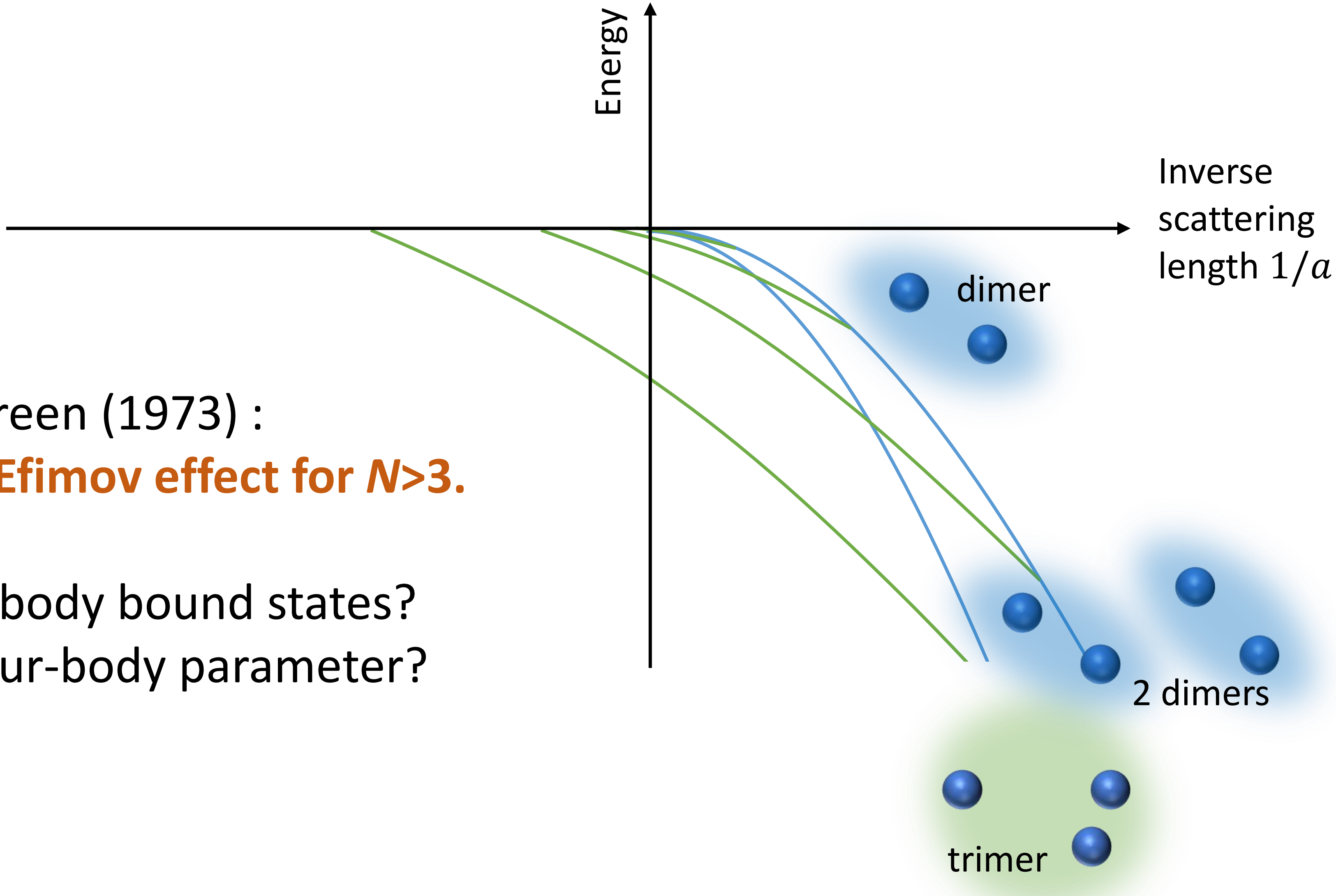
Observations in ultra-cold atomic gases

Observation with three distinguishable states of lithium atoms
(Heidelberg, Tokyo)



Science, 330, 940, Lompe et al (2010)

Four bosons



Amado & Green (1973) :
No *N*-body Efimov effect for *N*>3.

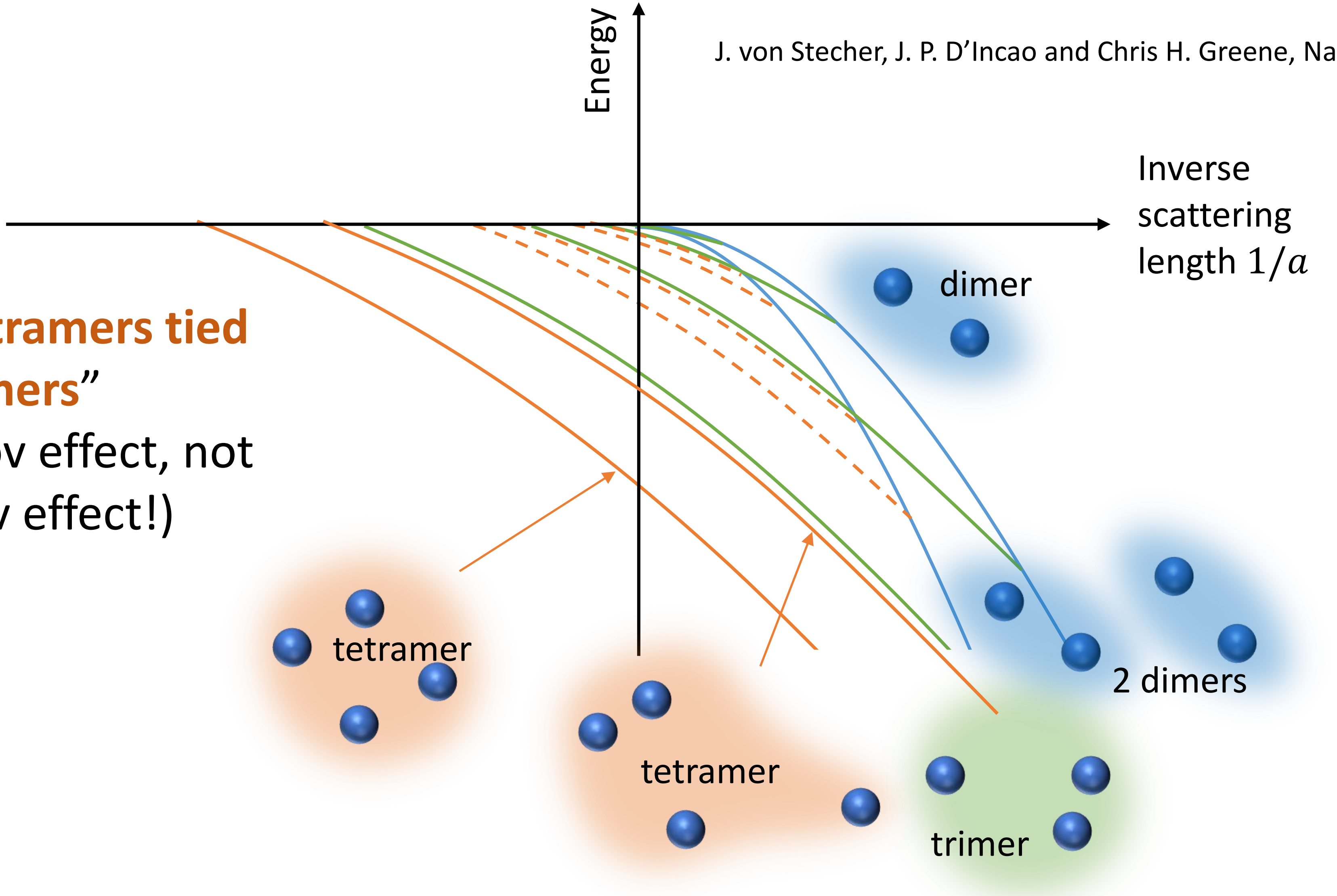
Universal 4-body bound states?
Is there a four-body parameter?

Four bosons

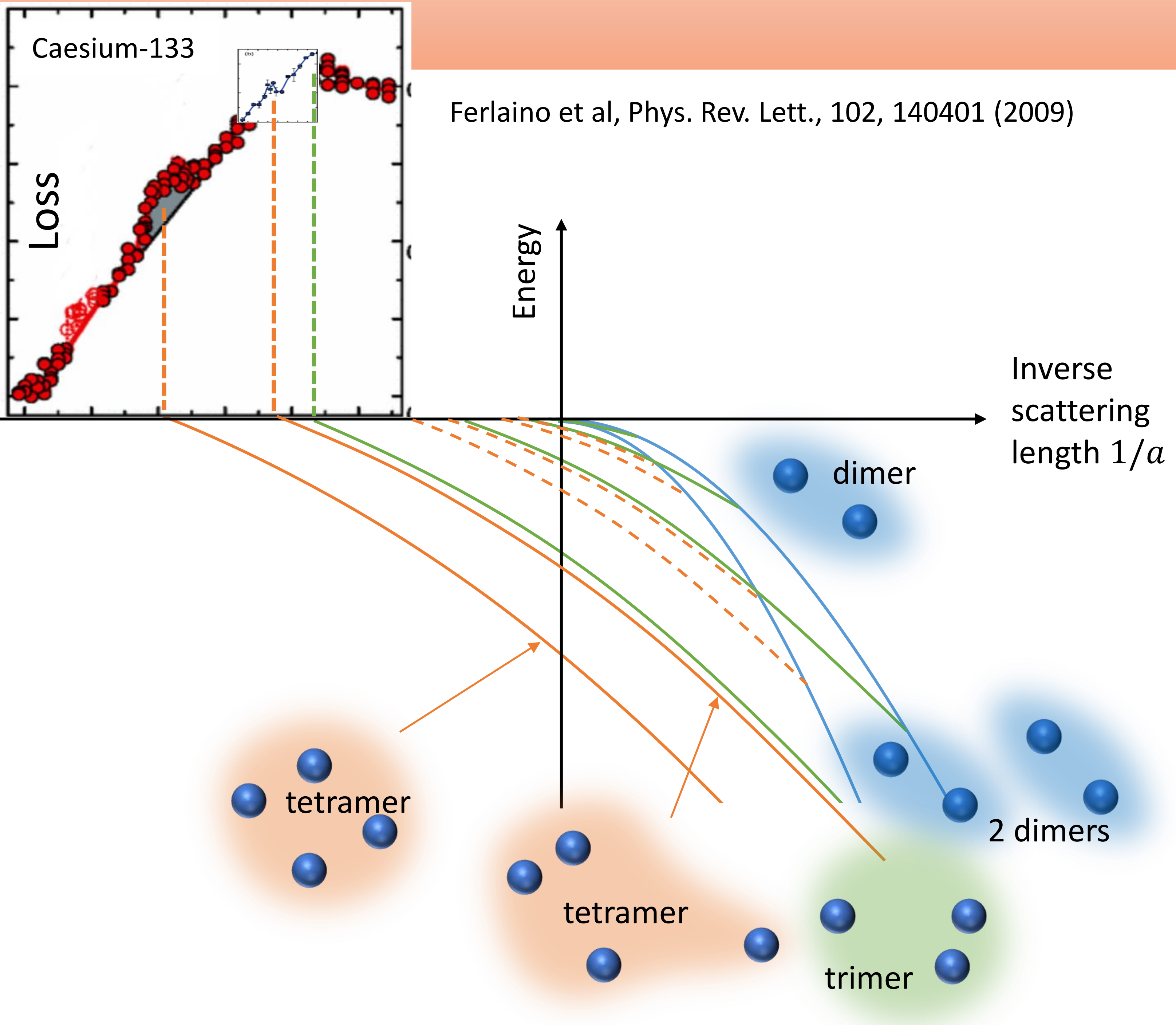
H-W Hammer L. Platter, Eur Phys J A 32, 113 (2007)

J. von Stecher, J. P. D’Incao and Chris H. Greene, Nat Phys 5, 417 (2009)

“**Universal tetramers tied to Efimov trimers**”
(3-body Efimov effect, not 4-body Efimov effect!)



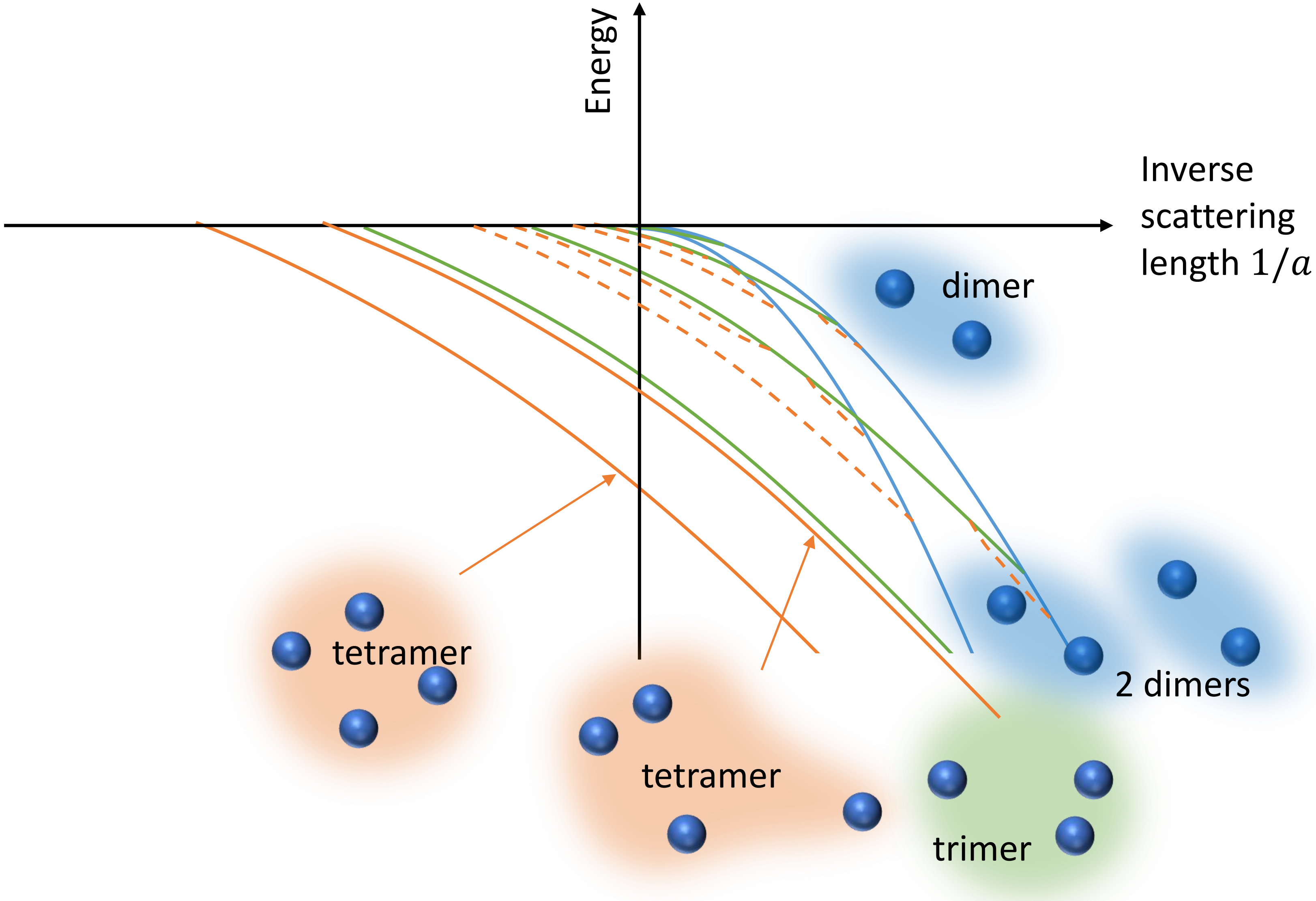
F



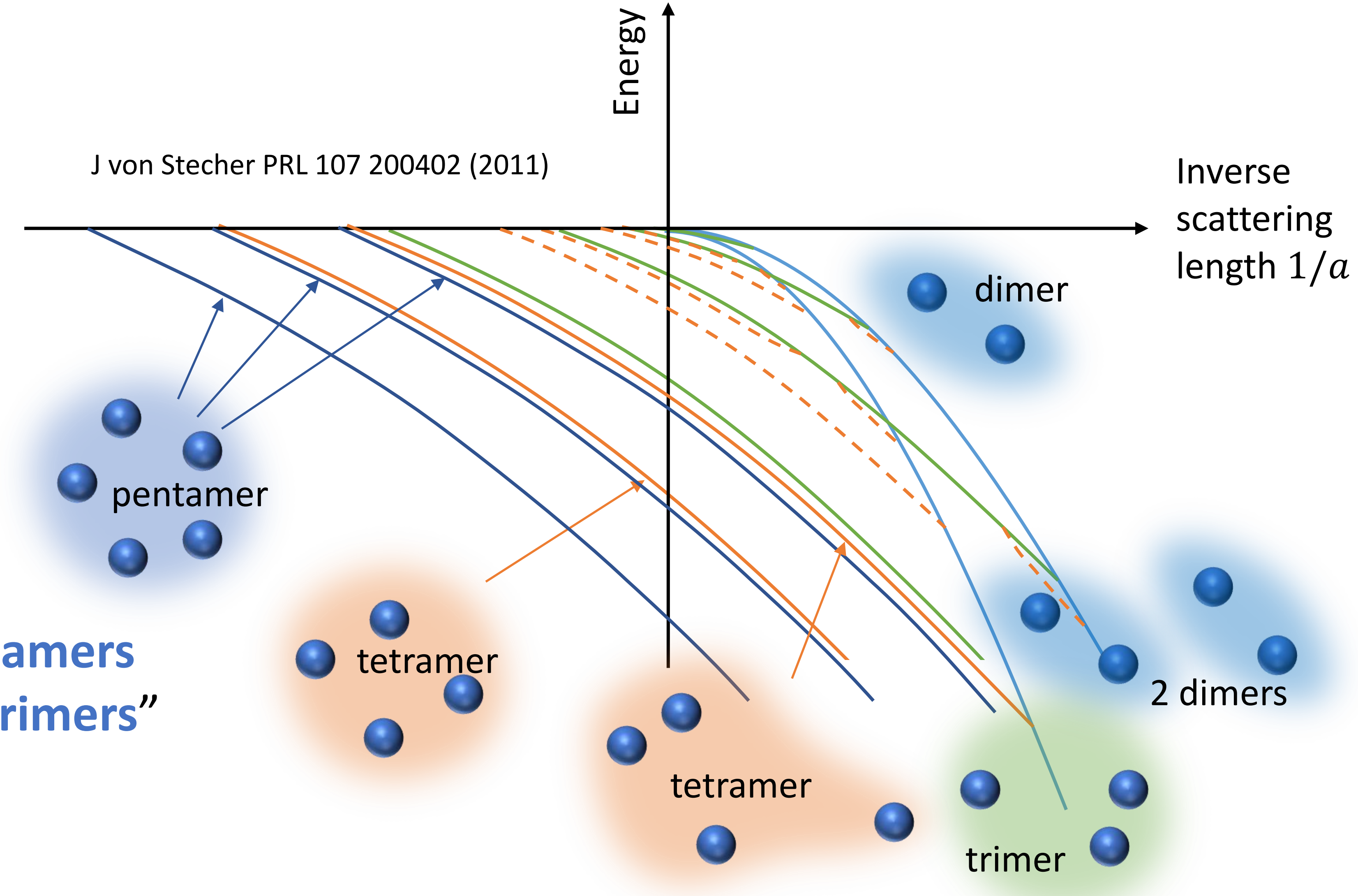
Ferlaino et al, Phys. Rev. Lett., 102, 140401 (2009)

Four bosons

A. Deltuva, Eur Phys Lett 95 43002 (2011)



More than 4 bosons



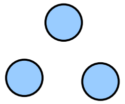
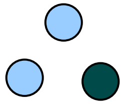
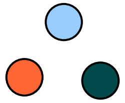
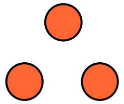
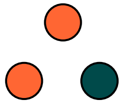
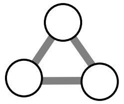
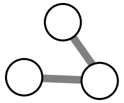
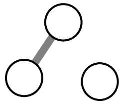
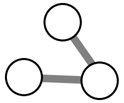
J von Stecher PRL 107 200402 (2011)

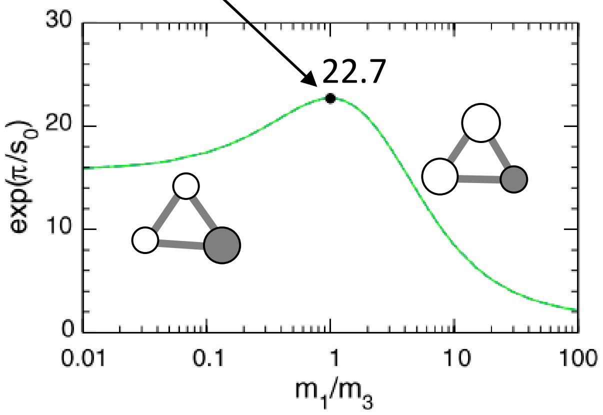
“Universal pentamers tied to Efimov trimers”

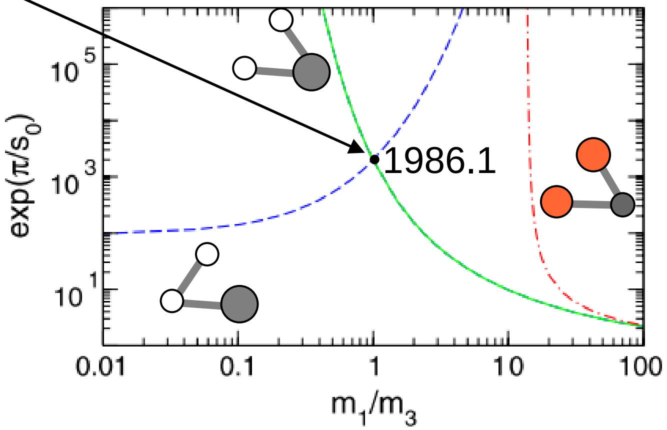
Mixtures

Particles of different statistics (bosons, fermions)
Particles of different masses
Particles in different spin states

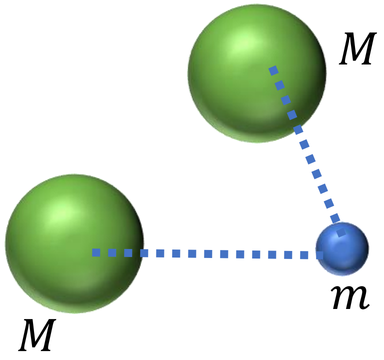
4. Overview 〉 Mixtures 〉 Bosons and fermions (spinless)

	 3 Bosons	 2 Bosons + 1 particle	 3 Distinguishable	 3 Fermions	 2 Fermions + 1 particle
Efimov effect	Number of resonant pairs  Yes	 Yes	 No	No	 Yes For mass ratio > **13.607**
Scaling factor $e^{\pi/\lvert s_0 \rvert}$ for equal masses	22.7	1986.1			
Scaling factor $e^{\pi/\lvert s_0 \rvert}$ for unequal masses					



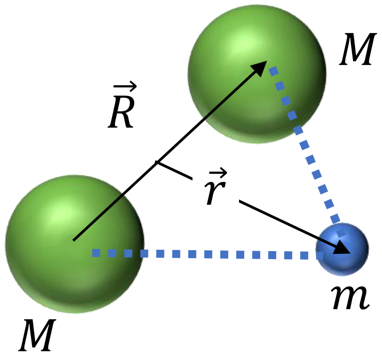


1 particle interacting with 2 identical particles



(no interaction between the two identical particles)

The Born-Oppenheimer approximation



$$\Psi\left(\vec{R},\vec{r}\right) = F\left(\vec{R}\right)\phi(\vec{r};\vec{R})$$ (like an electron with two nuclei)

Solve the motion of the light particle in presence of the 2 heavy particles:

$$-\frac{\hbar^2}{2m}\nabla_r^2\phi\left(\vec{r};\vec{R}\right) = \underbrace{-\frac{\hbar^2\kappa(R)^2}{2m}}_{\text{Born-Oppenheimer potential}}\phi(\vec{r};\vec{R})$$ and the two-body conditions

Lowest energy solution:

$$\phi\left(\vec{r};\vec{R}\right) = \frac{\exp\left(-\kappa\left|\vec{r}-\vec{R}/2\right|\right)}{|\vec{r}-\frac{\vec{R}}{2}|} + \frac{\exp\left(-\kappa|\vec{r}+\vec{R}/2|\right)}{|\vec{r}+\frac{\vec{R}}{2}|}$$

$$\kappa - \frac{e^{-\kappa R}}{R} = 1/a \quad \xrightarrow{a\to\infty} \quad \boxed{\kappa(R) = \frac{\Omega}{R}}$$

$$\Omega = W(1) \approx 0.567$$

Motion of the heavy particles:

$$\left(-\frac{\hbar^2}{M}\nabla_R^2 - \frac{\hbar^2\kappa(R)^2}{2m}\right)F\left(\vec{R}\right) = EF(\vec{R})$$

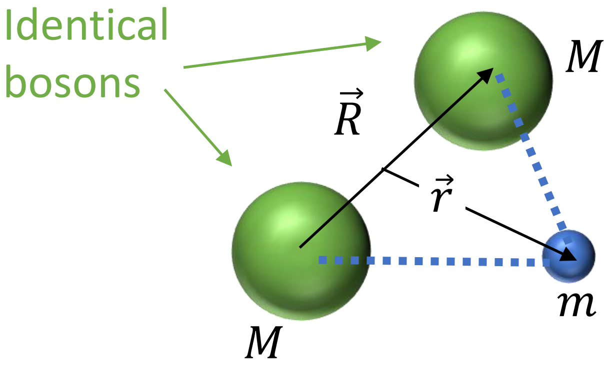
$$\Big\downarrow \qquad\qquad \Big\downarrow$$

For a certain angular momentum L between the two heavy particles:

$$F\left(\vec{R}\right) = \frac{u_L(R)}{R}P_L(\cos\theta)$$

$$\left(-\frac{d^2}{dR^2} + \frac{L(L+1)}{R^2} - \frac{M}{2m}\frac{\Omega^2}{R^2}\right)u_L(R) = \frac{ME}{\hbar^2}u_L(R)$$

The Born-Oppenheimer approximation



Competition

Centrifugal repulsion

Efimov attraction

$$\left(-\frac{d^2}{dR^2}+\underbrace{\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}}_{V_0(r)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}\right)u_L(R) \quad = \frac{ME}{\hbar^2}u_L(R)$$

$$|s_0|^2=\frac{M}{2m}\Omega^2-L(L+1)-\frac{1}{4}$$

For $L=0$

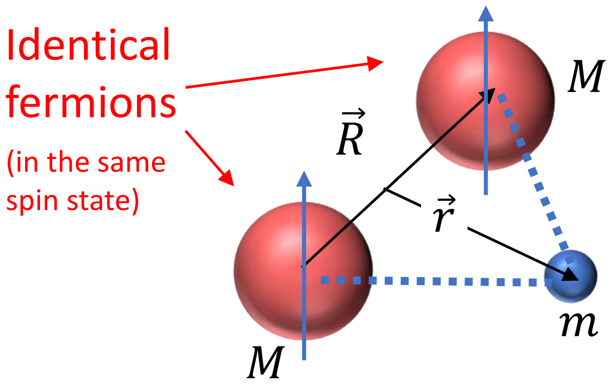
No centrifugal repulsion. Efimov attraction wins. ➡

Efimov effect

physics depend on a and Λ

Stronger for large mass ratio M/m

The Born-Oppenheimer approximation



Competition

Centrifugal repulsion

Efimov attraction

$$\left(-\frac{d^2}{dR^2}+\underbrace{\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}}_{V_0(r)=-\frac{|s_0|^2+\frac{1}{4}}{R^2}}\right)u_L(R) = \frac{ME}{\hbar^2}u_L(R)$$

$$|s_0|^2 = \frac{M}{2m}\Omega^2 - L(L+1) - \frac{1}{4}$$

For $L = 1$

Critical mass ratio such that $|s_0| = 0$: $\frac{M}{m} = 13.990296\ldots$ (exact result: $13.607\ldots$)

$\frac{M}{m} > 13.6$: Efimov attraction wins.

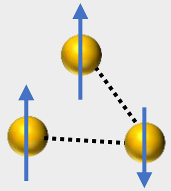
Efimov effect
physics depend on a and Λ

$\frac{M}{m} < 13.6$: Efimov attraction loses.

no Efimov effect
physics depend only on a

Particular case

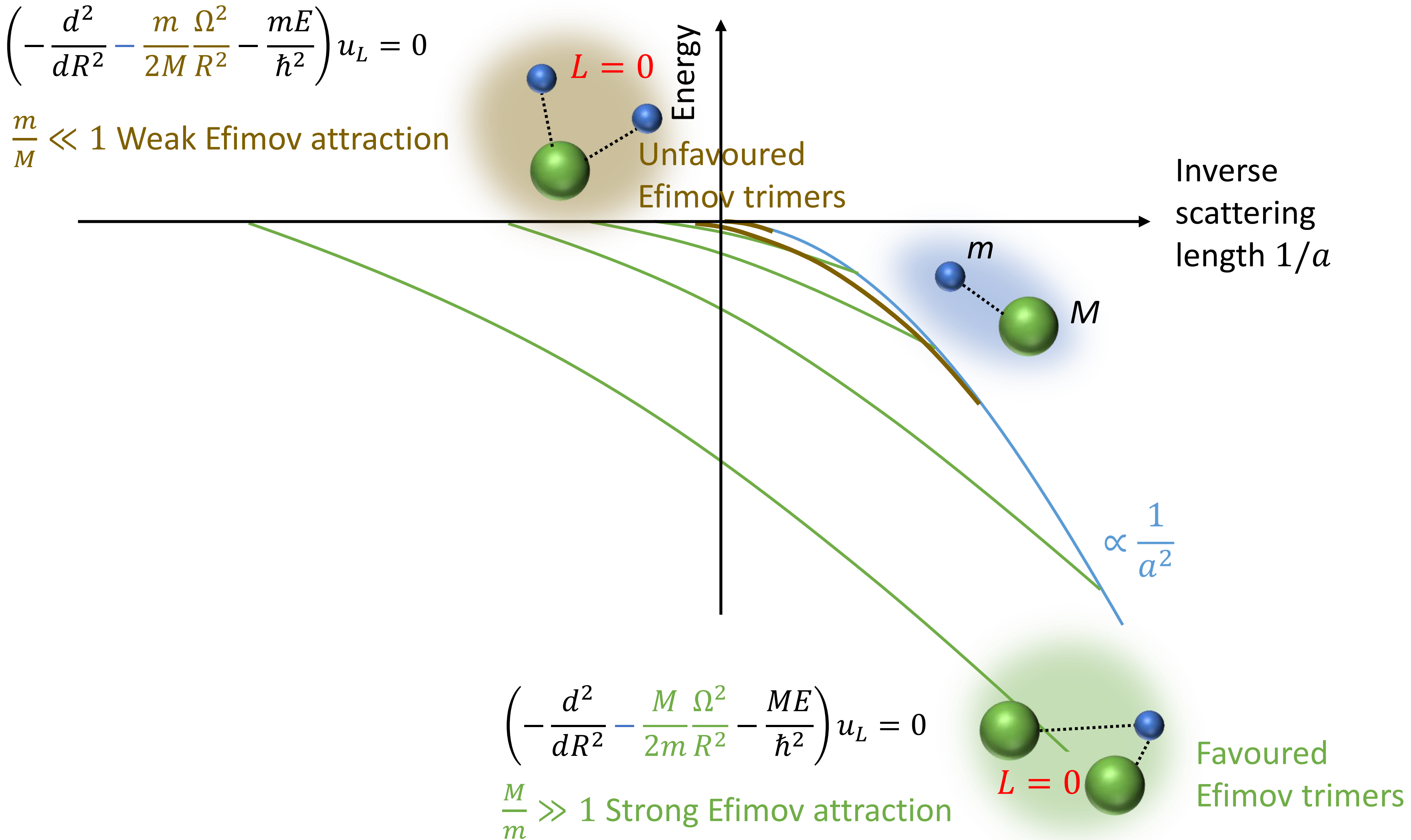
No Efimov effect for 3 fermions with spin 1/2



$\frac{M}{m} = 1$

e.g. no trineutron

Mixtures of two kinds of bosons



$$\left(-\frac{d^2}{dR^2}-\frac{m}{2M}\frac{\Omega^2}{R^2}-\frac{mE}{\hbar^2}\right)u_L=0$$

$\frac{m}{M}\ll 1$ Weak Efimov attraction

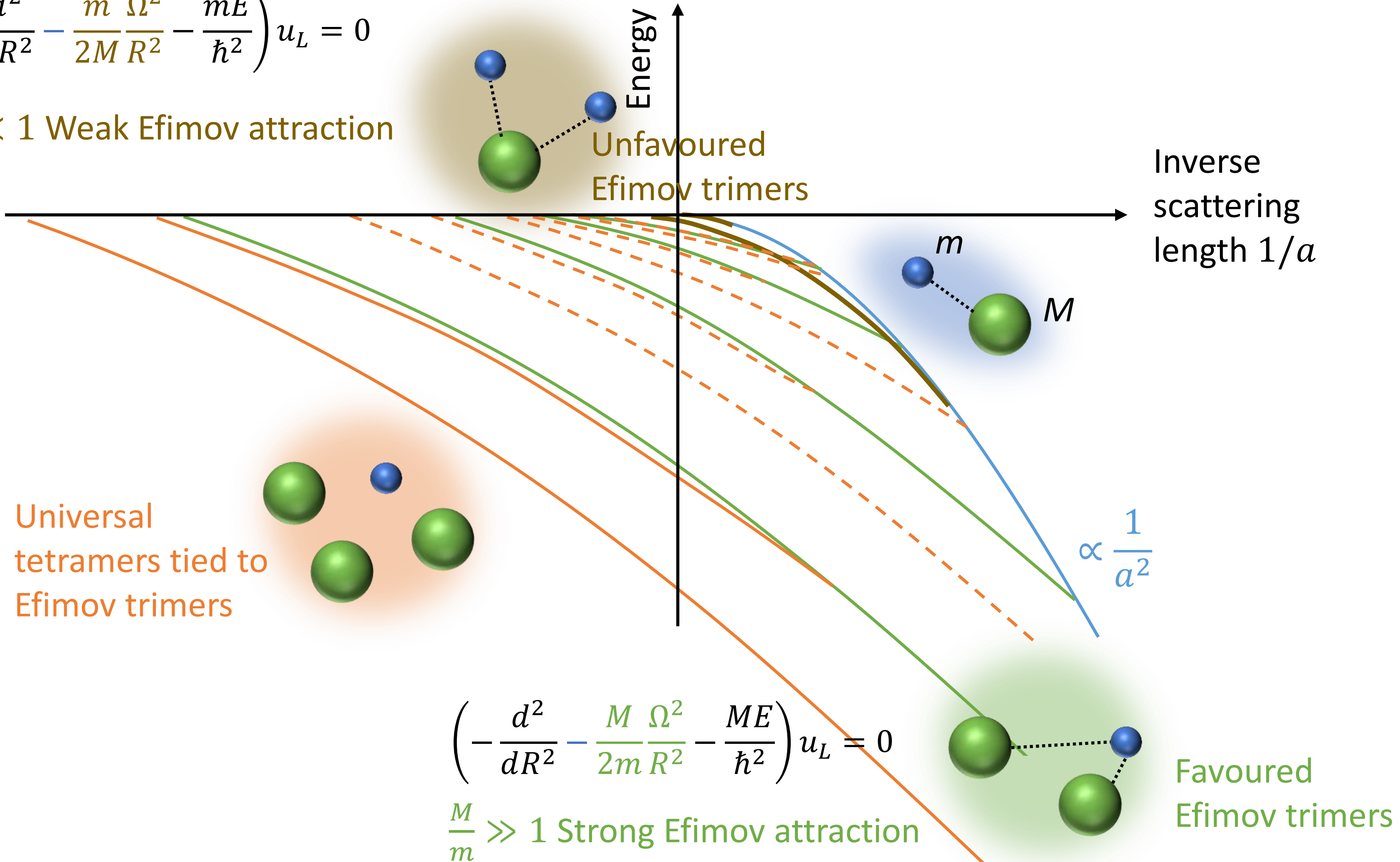
$$\left(-\frac{d^2}{dR^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}-\frac{ME}{\hbar^2}\right)u_L=0$$

$\frac{M}{m}\gg 1$ Strong Efimov attraction

Mixtures of two kinds of bosons

$$\left(-\frac{d^2}{dR^2} - \frac{m}{2M}\frac{\Omega^2}{R^2} - \frac{mE}{\hbar^2}\right)u_L = 0$$

$\frac{m}{M} \ll 1$ Weak Efimov attraction



Unfavoured Efimov trimers

Universal tetramers tied to Efimov trimers

Favoured Efimov trimers

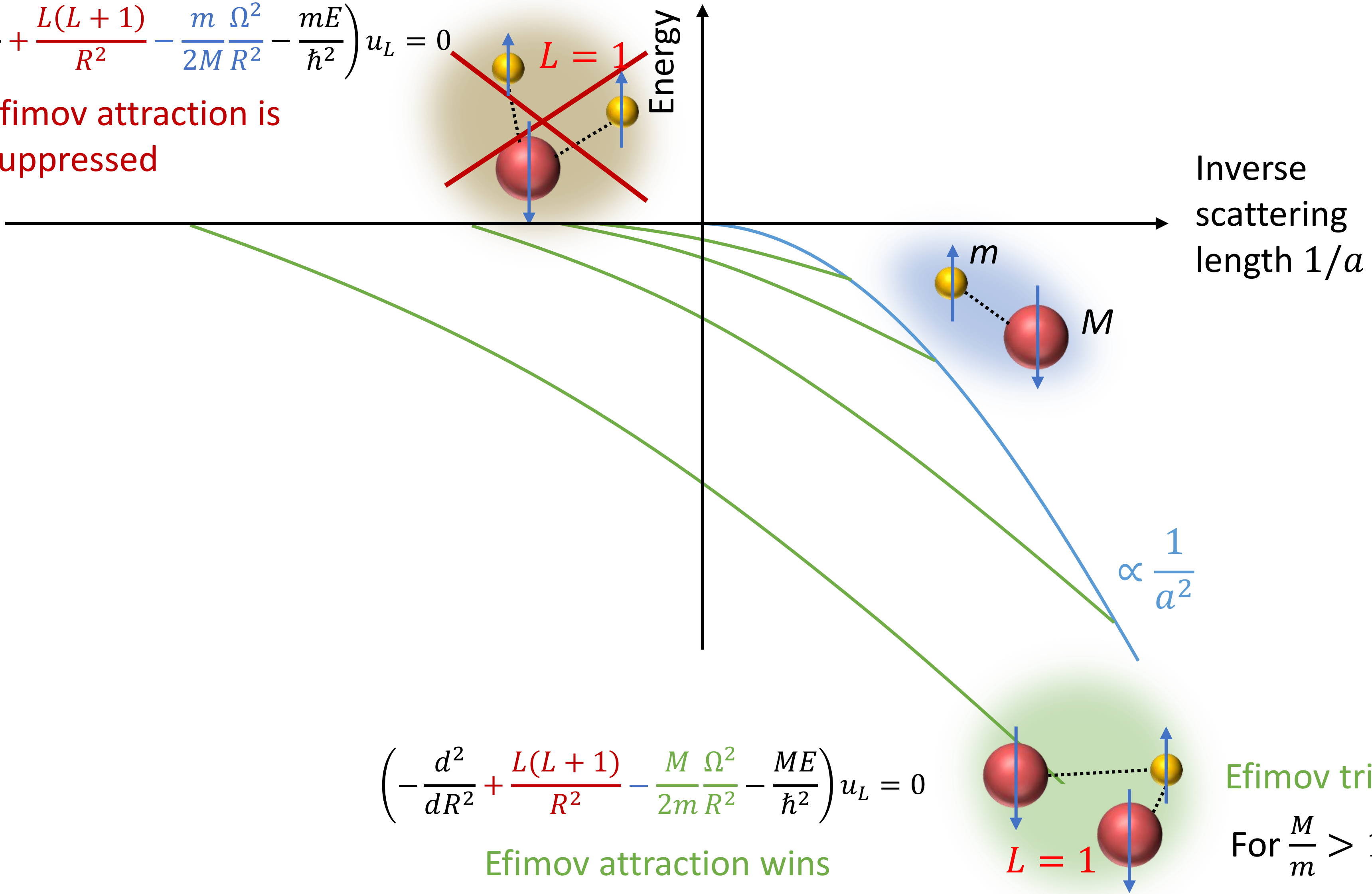
$$\left(-\frac{d^2}{dR^2} - \frac{M}{2m}\frac{\Omega^2}{R^2} - \frac{ME}{\hbar^2}\right)u_L = 0$$

$\frac{M}{m} \gg 1$ Strong Efimov attraction

Mixtures of two kinds of fermions (polarised = spinless)

$$\left(-\frac{d^2}{dR^2}+\frac{L(L+1)}{R^2}-\frac{m}{2M}\frac{\Omega^2}{R^2}-\frac{mE}{\hbar^2}\right)u_L=0$$

Efimov attraction is suppressed



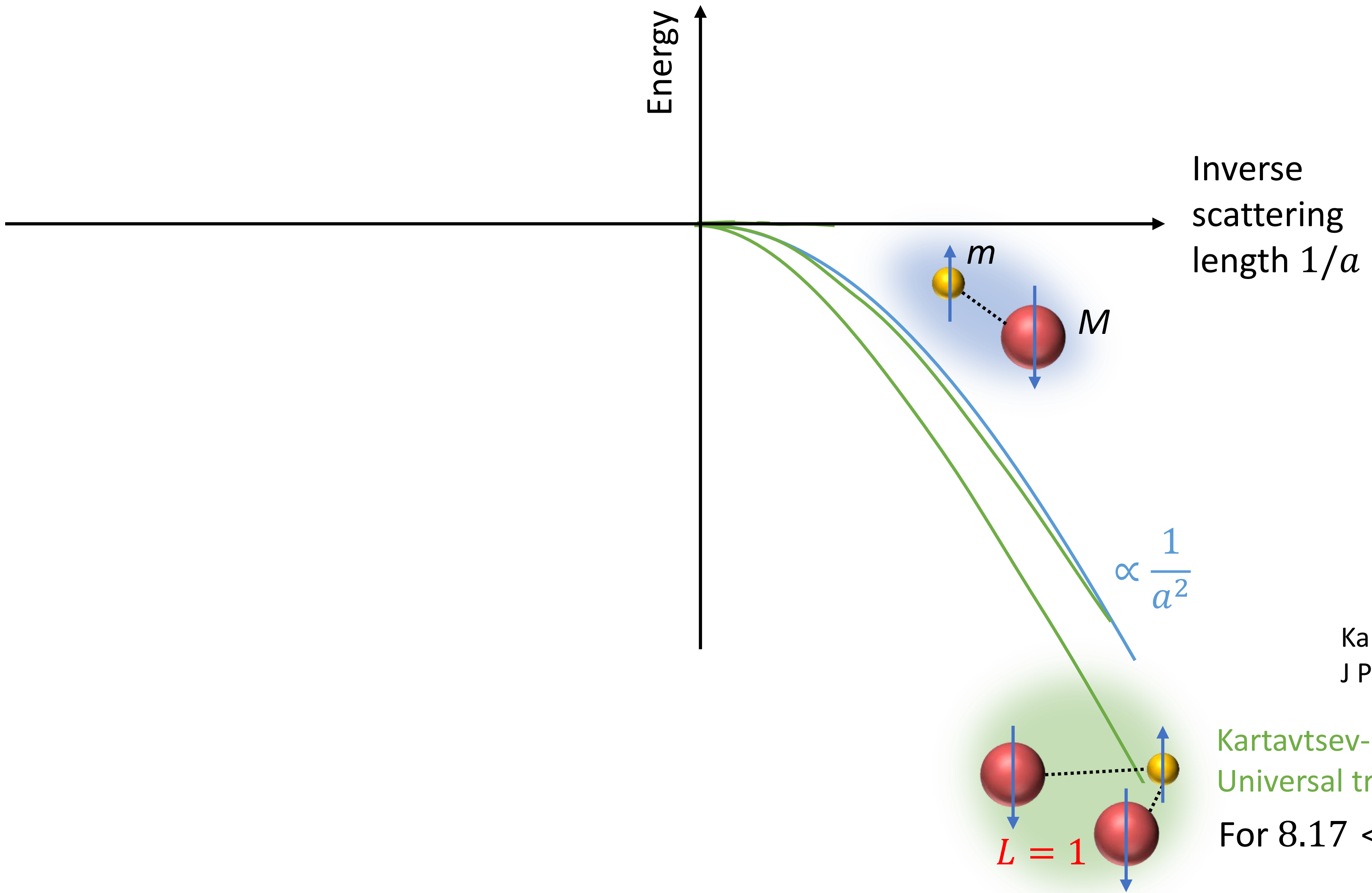
$$\left(-\frac{d^2}{dR^2}+\frac{L(L+1)}{R^2}-\frac{M}{2m}\frac{\Omega^2}{R^2}-\frac{ME}{\hbar^2}\right)u_L=0$$

Efimov attraction wins

Efimov trimers

For $\frac{M}{m}>13.6$

Mixtures of two kinds of fermions (polarised = spinless)

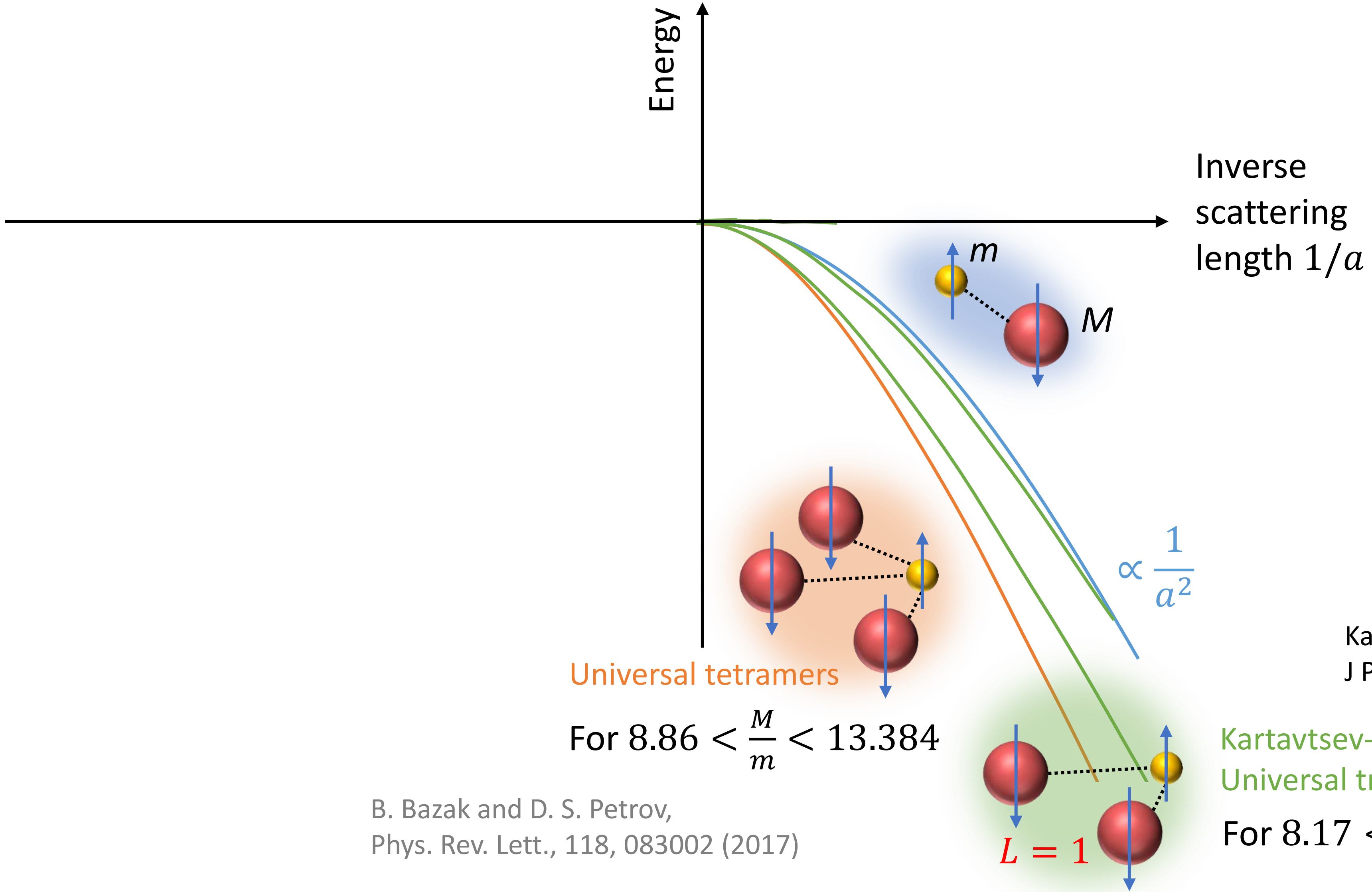


Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Kartavtsev-Malykh
Universal trimers

For $8.17 < \frac{M}{m} < 13.6$

Mixtures of two kinds of fermions (polarised = spinless)



Universal tetramers

For $8.86 < \frac{M}{m} < 13.384$

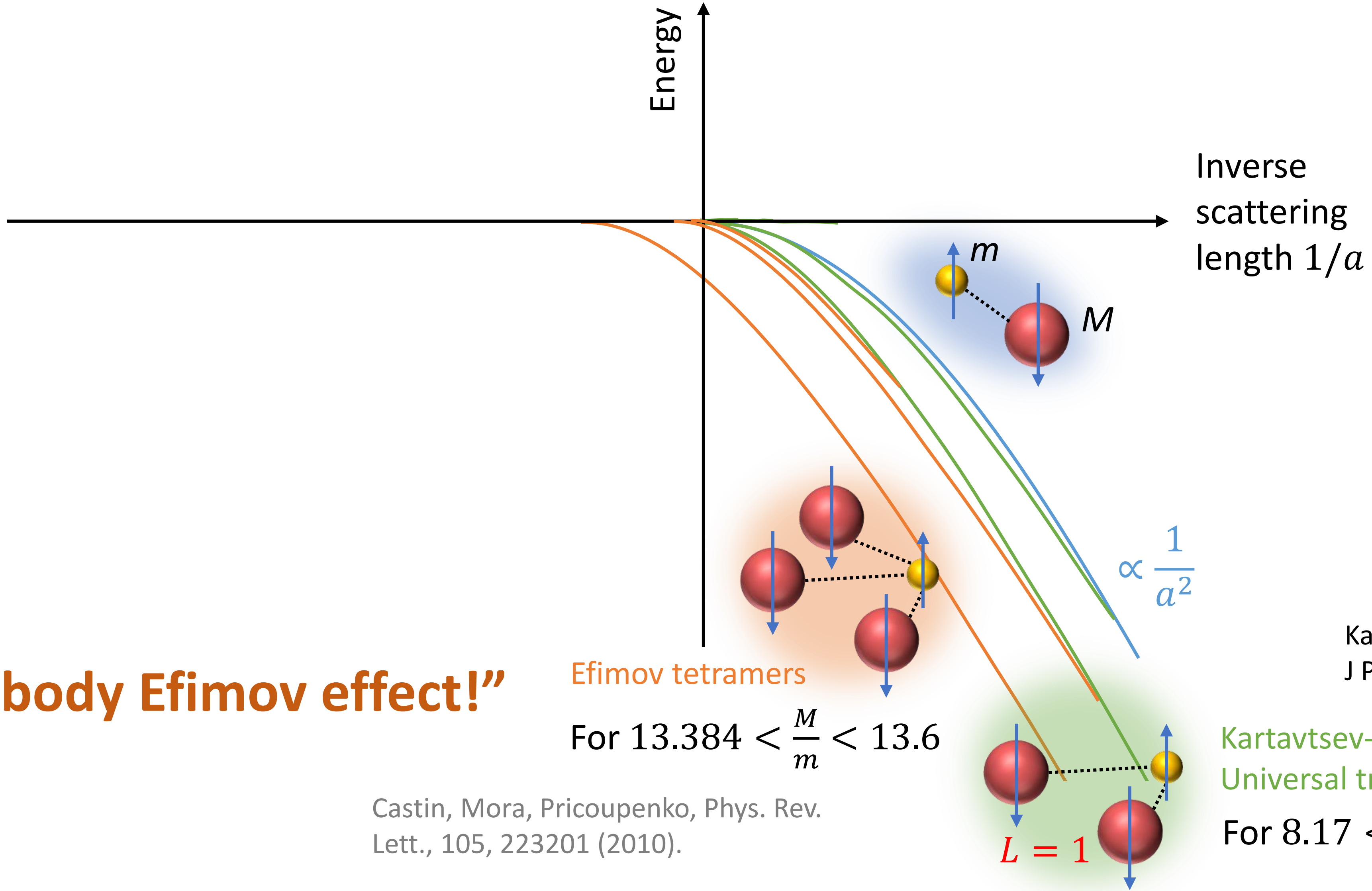
B. Bazak and D. S. Petrov,
Phys. Rev. Lett., 118, 083002 (2017)

Kartavtsev-Malykh
Universal trimers

For $8.17 < \frac{M}{m} < 13.6$

Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Mixtures of two kinds of fermions (polarised = spinless)



"4-body Efimov effect!"

Efimov tetramers

For $13.384 < \frac{M}{m} < 13.6$

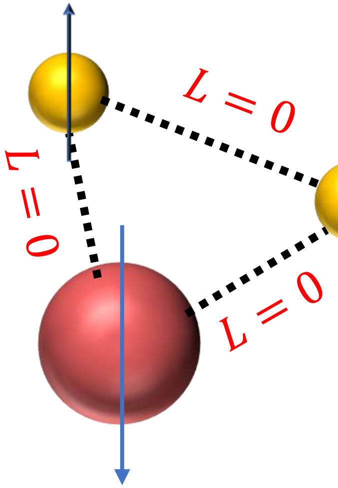
Castin, Mora, Pricoupenko, Phys. Rev. Lett., 105, 223201 (2010).

Kartavtsev-Malykh Universal trimers

For $8.17 < \frac{M}{m} < 13.6$

Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

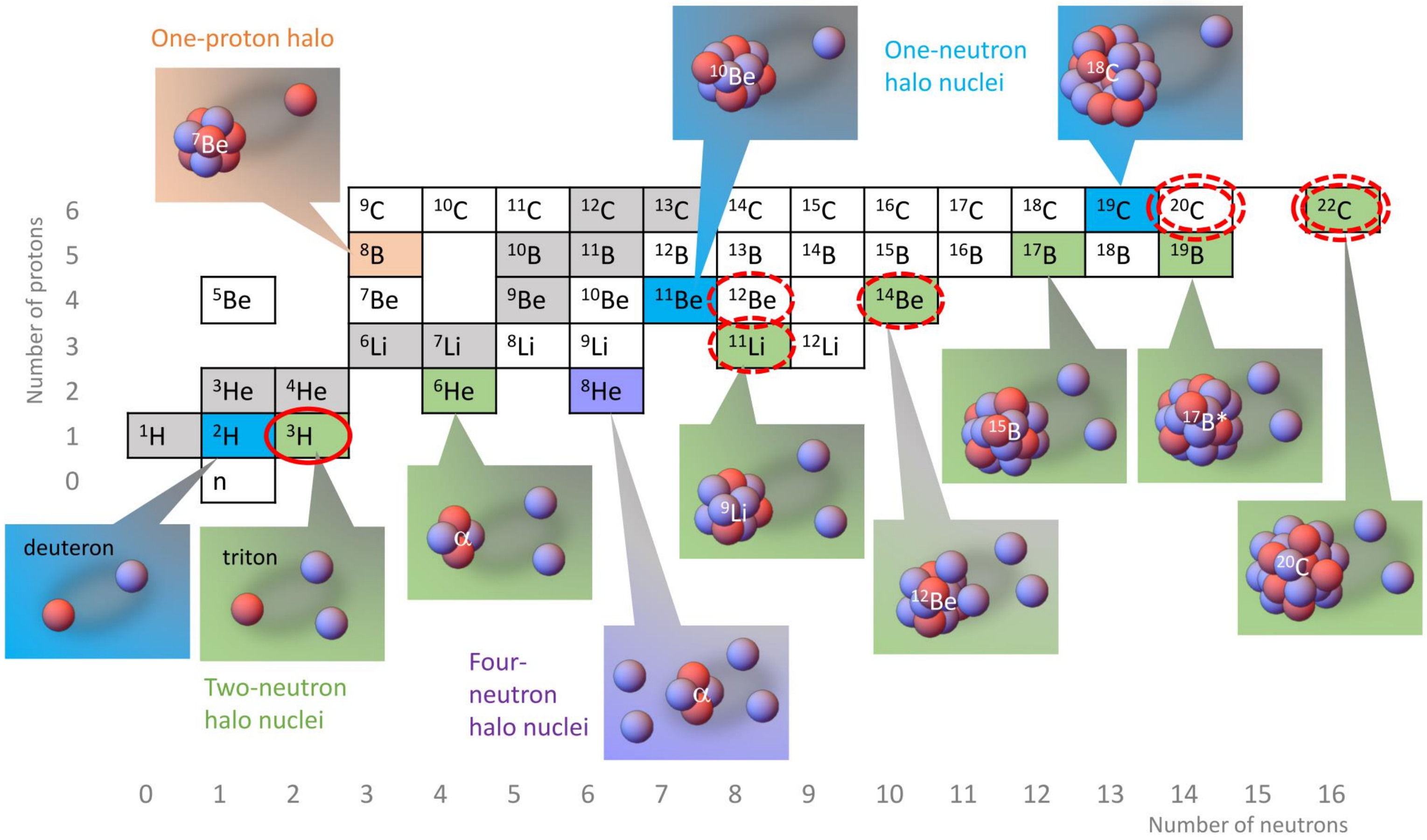
Two kinds of fermions with spin



→ Two-neutron halo nuclei?

(All pairs can interact in the s wave)

Halo nuclei



Efimov trimer candidates:

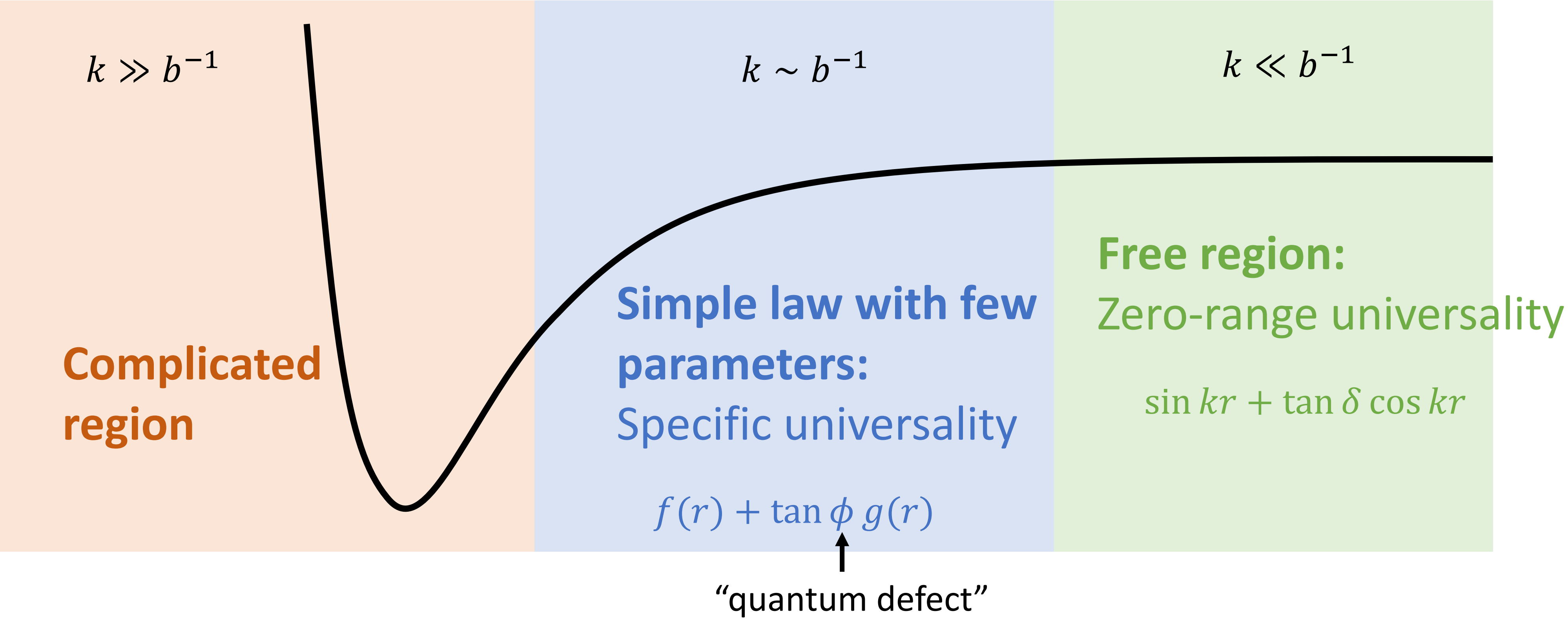
◌ Ground state

◎ First excited state

(to be confirmed)

Van der Waals universality

Other classes of universalities



$k \gg b^{-1}$

Complicated region

$k \sim b^{-1}$

Simple law with few parameters:

Specific universality

$$f(r) + \tan\phi\, g(r)$$

"quantum defect"

$k \ll b^{-1}$

Free region:

Zero-range universality

$$\sin kr + \tan\delta \cos kr$$

Examples:

Electron in a Rydberg atom $V(r) = \frac{q^2}{4\pi\epsilon_0}\frac{1}{r}$ Bohr radius: $a_C = \frac{4\pi\epsilon_0}{q^2}\frac{\hbar^2}{2\mu}$

Two neutral atoms $V(r) = -C_6/r^6$ Van der Waals length: $l_{vdW} = \frac{1}{2}\left(\frac{2\mu C_6}{\hbar^2}\right)^{1/4}$

5. Van der Waals universality

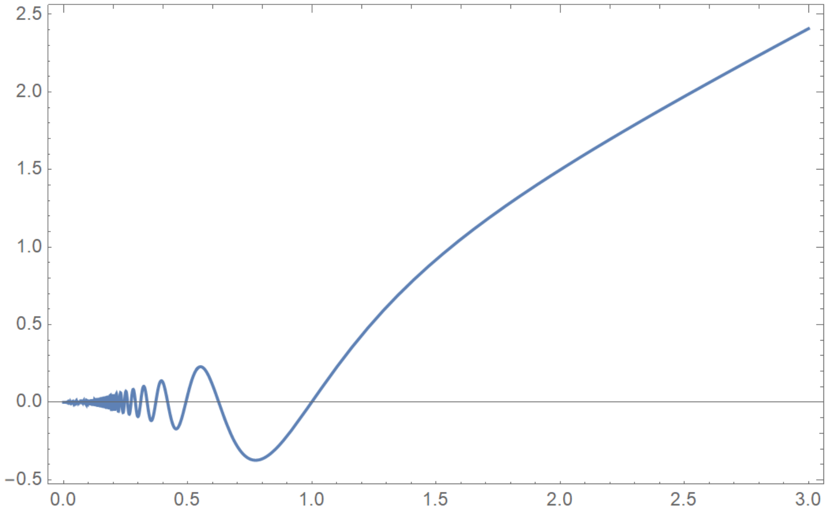
Solution of $-C_6/r^6$ potential at zero energy

$$u(r) = f(r) - \frac{a}{l_{vdW}} g(r)$$

Van der Waals length

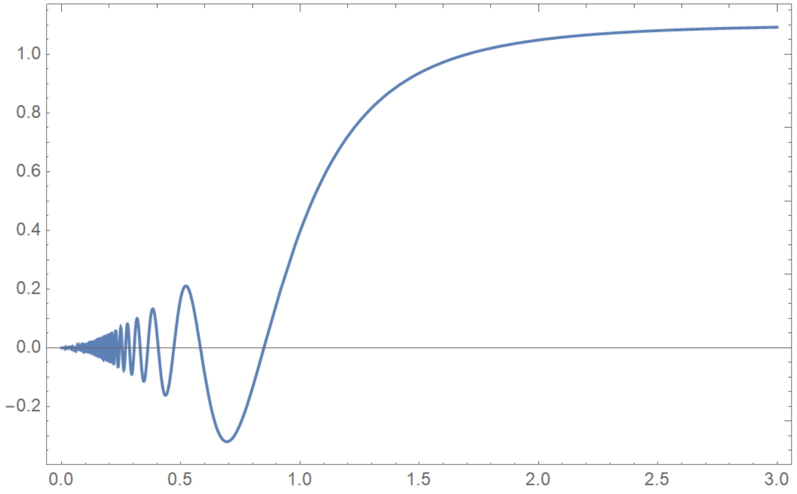
$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

$$f(r) = \Gamma(3/4)\sqrt{x}J_{-\frac{1}{4}}(2x^{-2})$$



$$x = r/\ell_{vdW}$$

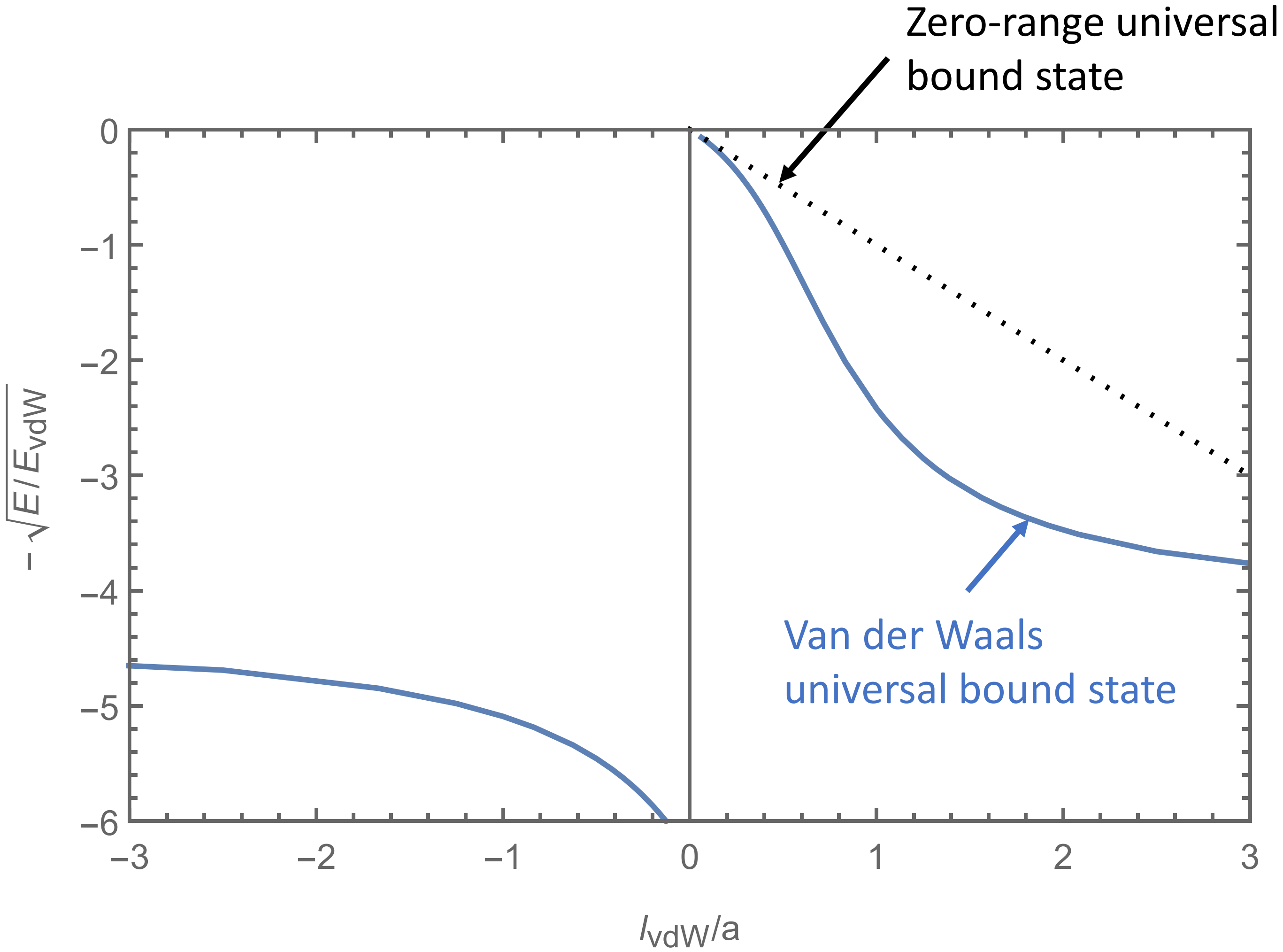
$$g(r) = \Gamma(5/4)\sqrt{x}J_{\frac{1}{4}}(2x^{-2})$$



$$x = r/\ell_{vdW}$$

5. Van der Waals universality

Solution of $-C_6/r^6$ potential at negative energy



Zero-range universal bound state

Van der Waals universal bound state

Van der Waals length

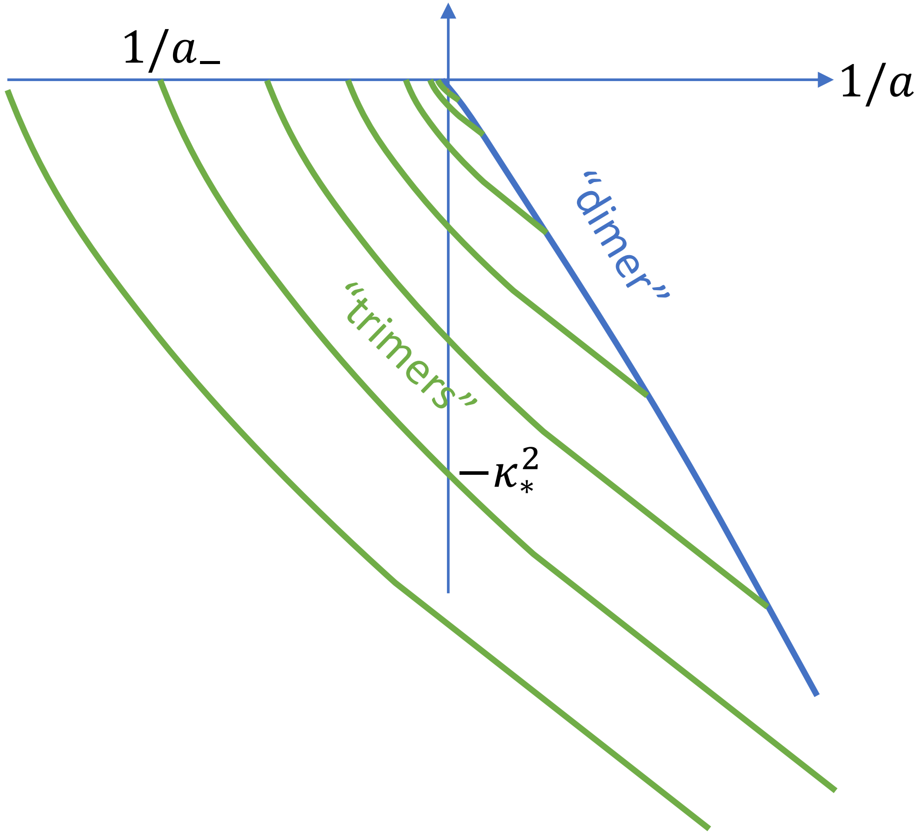
$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

Van der Waals energy scale

$$E_{vdW} = \frac{\hbar^2}{2\mu l_{vdW}^2}$$

The three-body parameter

In the zero-range theory, the three-body has to be introduced "by hand" as an extra boundary condition to quantise the spectrum:

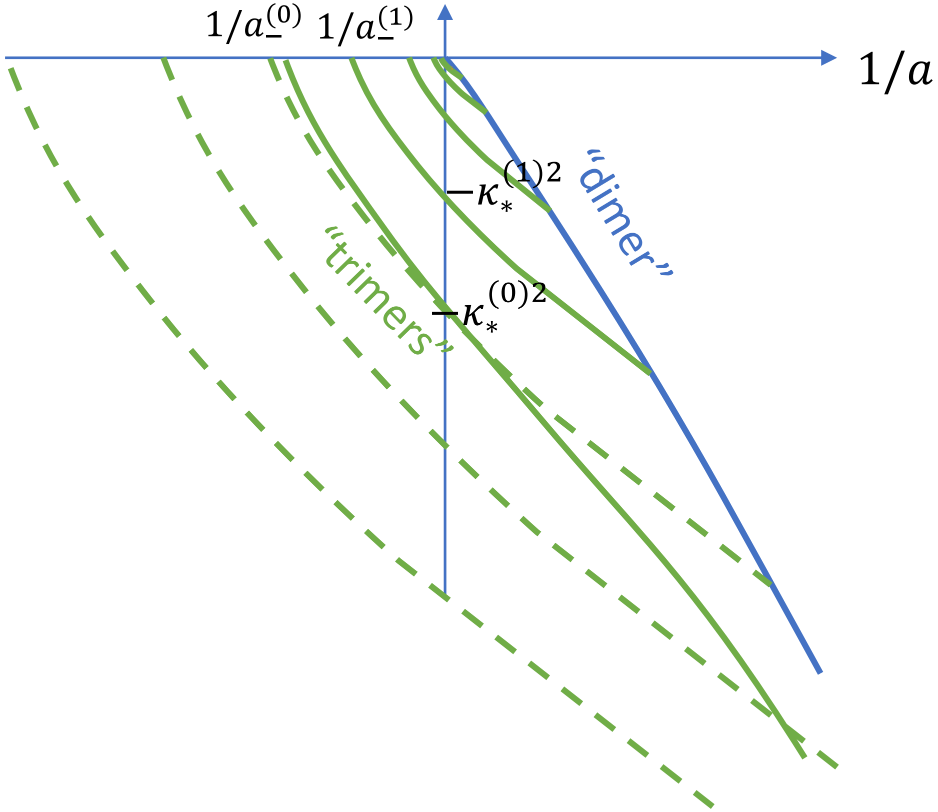


$$a_- = -1.50763/\kappa_*$$

The three-body parameter

For a system with finite-range interactions, the spectrum is bounded from below and the three-body parameter is set from the two-body or three-body interactions

$$\kappa_* = \lim_{n\to\infty} \kappa_*^{(n)} e^{-n\pi/|s_0|}$$

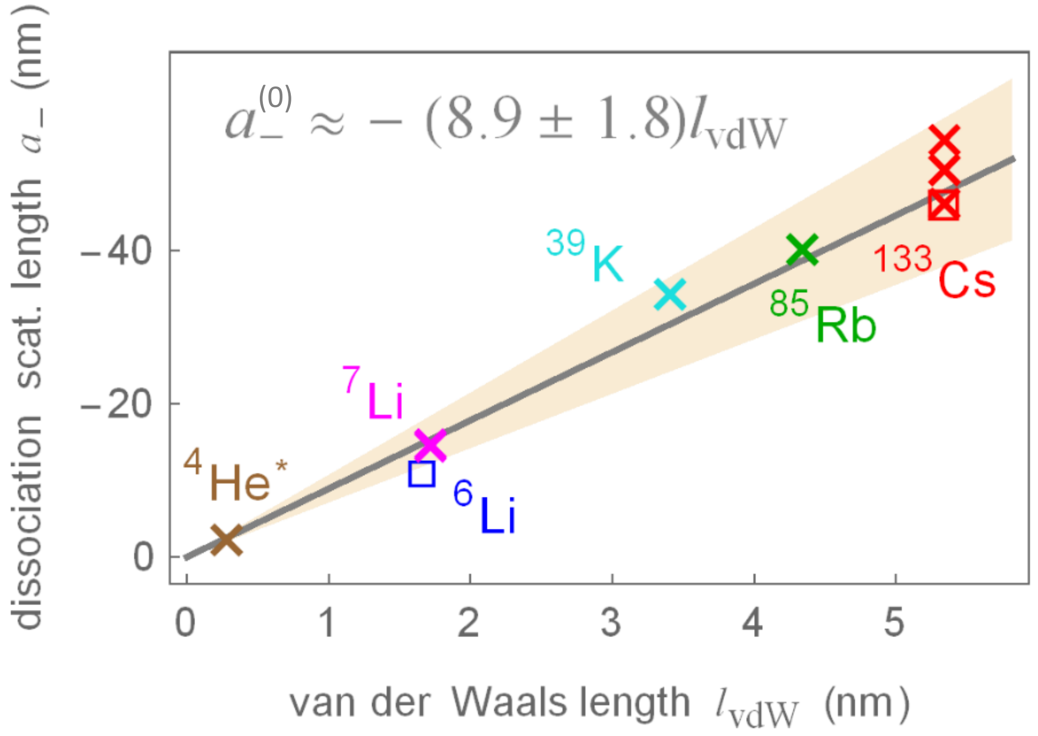


Question: what in the interaction potentials determines the value of the three-body parameter?

Range of two-body forces?
What about three-body forces?

The three-body parameter

Experiments with atoms: “**Van der Waals universality of the three-body parameter**”



The potential between two atoms has a van der Waals tail:

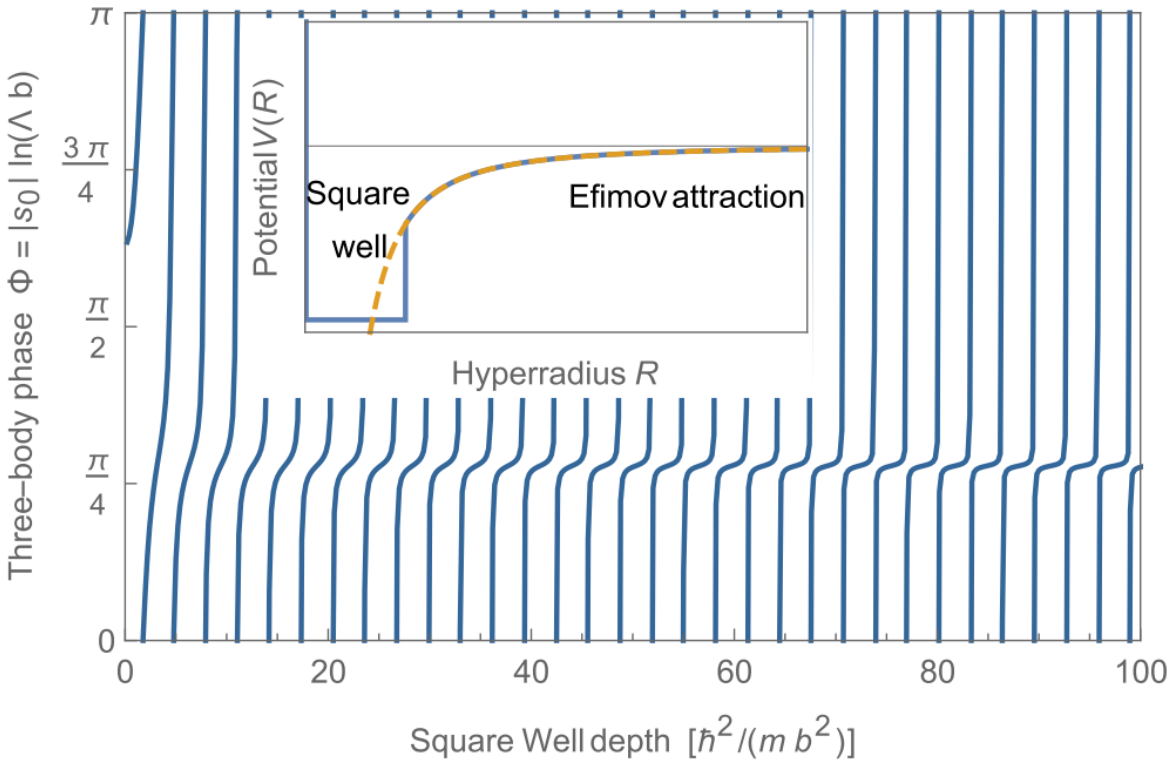
$$V(r) \xrightarrow[r\to\infty]{} -C_6/r^6$$

Van der Waals length

$$l_{vdW} = \frac{1}{2}\left(\frac{mC_6}{\hbar^2}\right)^{1/4}$$

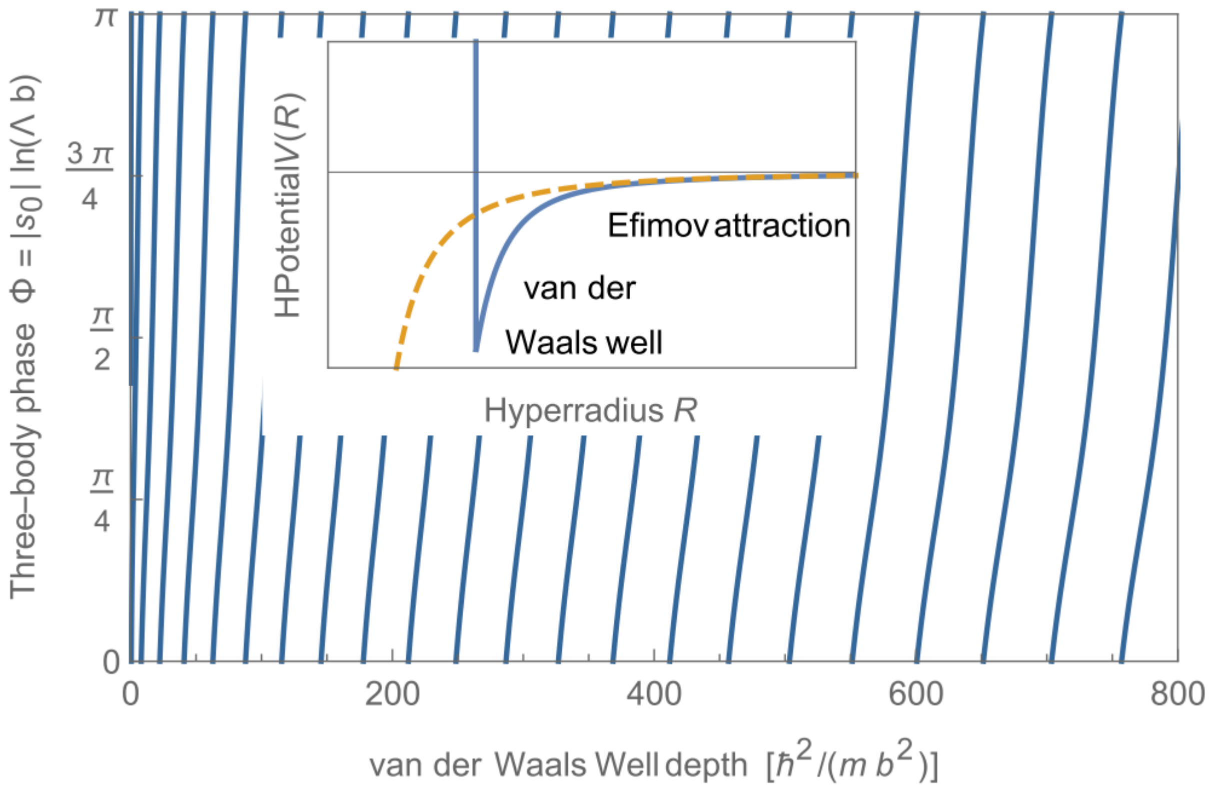
The three-body parameter

Is it quantum reflexion?



The three-body parameter

Is it quantum reflexion? No.



The three-body parameter

J. Wang, J. D’Incao, B. Esry, and C. Greene, “Origin of the Three-Body Parameter Universality in Efimov Physics.” Phys. Rev. Lett., 108, 263001 (2012).

Adiabatic Hyperspherical Representation:

$$\Psi = \sum_n F_n(R)\Phi_n(\Omega; R)$$

Hyper-radius $R^2 = \frac{2}{3}(r_{12}^2 + r_{23}^2 + r_{31}^2)$

$$\left(-\frac{d^2}{dR^2} + W_n(R) - E\right)F_n(R) + \sum_{n' \neq n} W_{n,n'}(R)F_{n'}(R) \ = 0$$

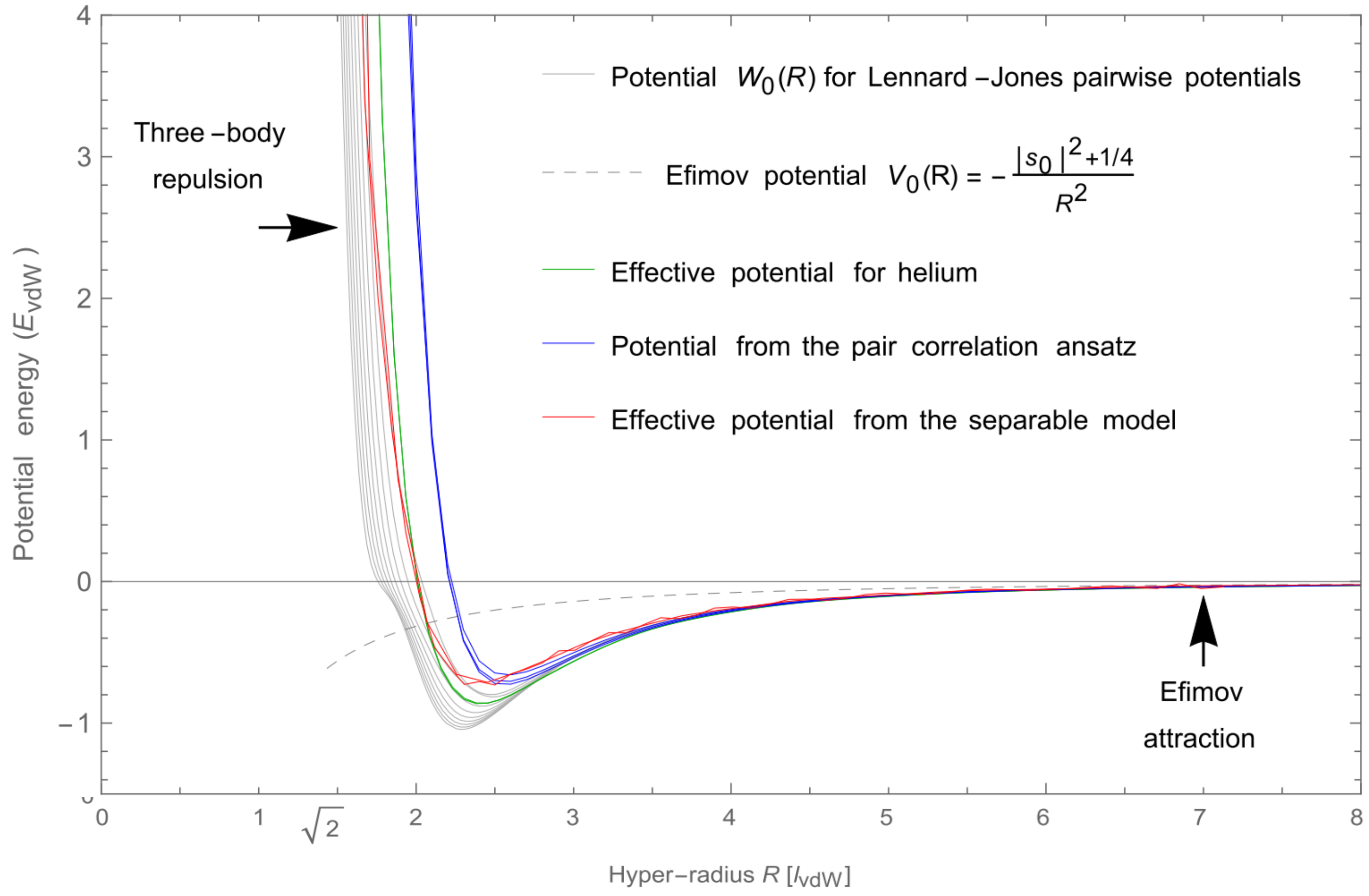
Solved for various two-body interactions with a van der Waals tail.
In the limit of deep van der Waals interactions:

$$\boxed{\kappa_*^{(0)} = (0.21 \pm 0.01)/\ell_{\mathrm{vdW}}}$$

$$\boxed{a_-^{(0)} = -(10.70 \pm 0.35)\,\ell_{\mathrm{vdW}}}$$

The three-body parameter

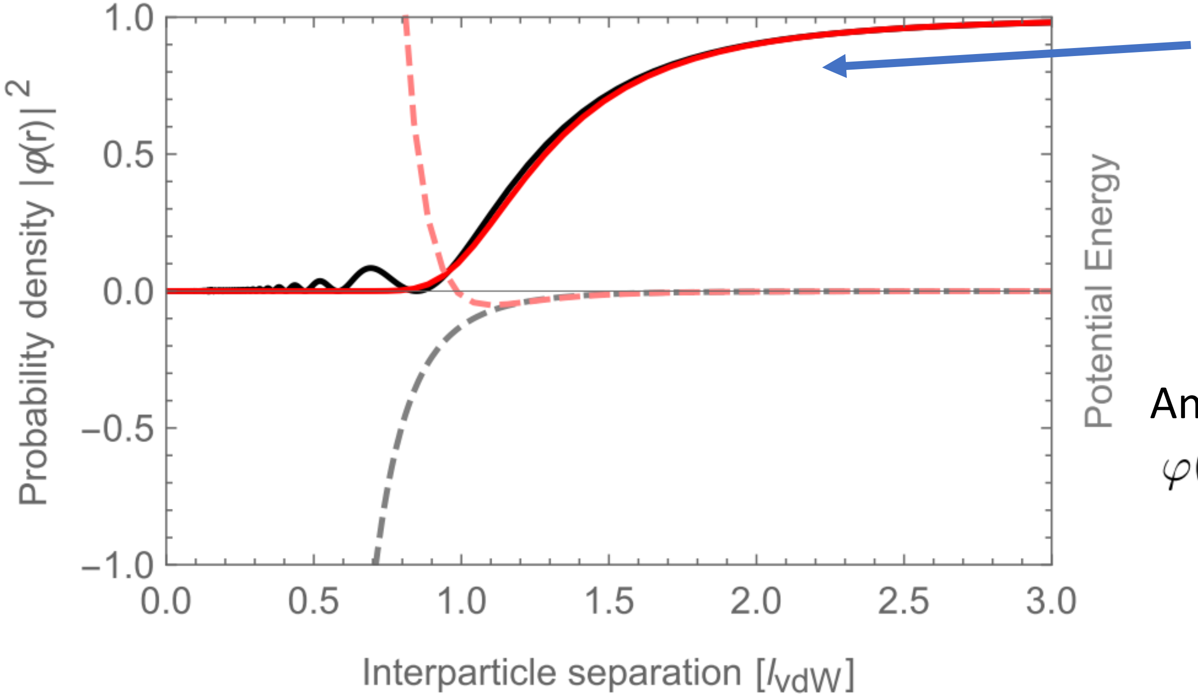
Three-body repulsive barrier



The three-body parameter

What is the origin of the three-body repulsion?
(how can we get repulsion from purely attractive interactions?)

Two-body van der Waals universality



Universal suppression of two-body probability for $r \lesssim l_{vdW}$

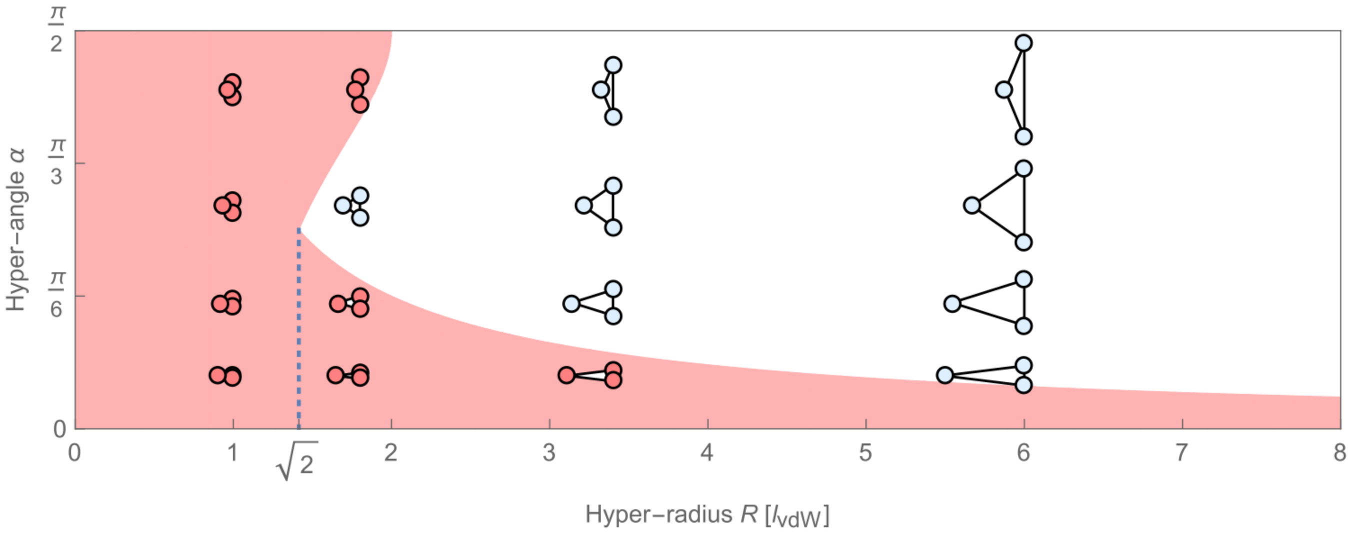
Analytical solution at zero energy

$$\varphi(r) \underset{r \gtrsim \ell_{\text{vdW}}}{=} \Gamma(5/4)\sqrt{x} J_{\frac{1}{4}}(2x^{-2})$$

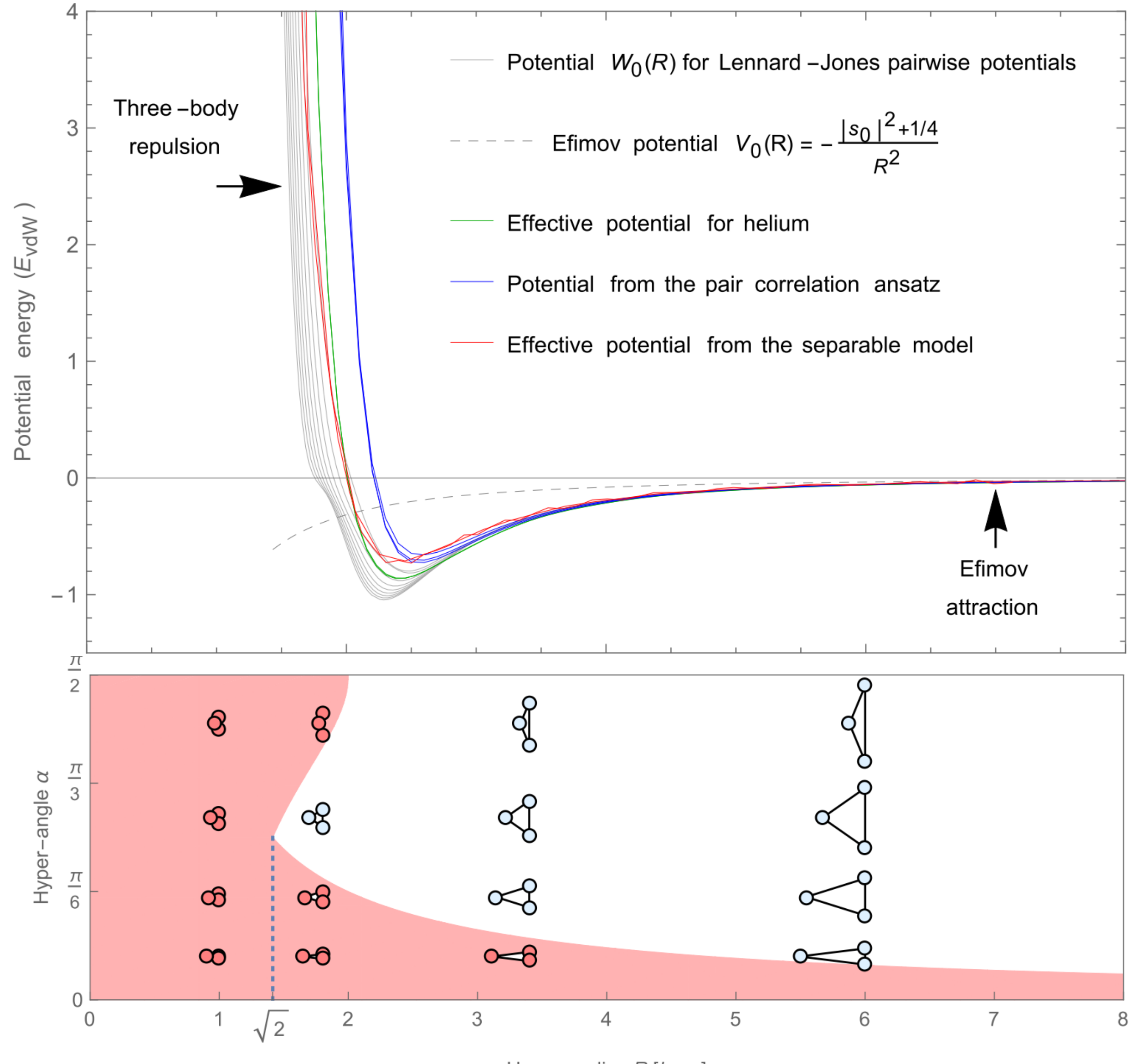
$$x = r/\ell_{\text{vdW}}$$

The three-body parameter

Consequence for three-body:
Suppressed configurations for $r_{ij} \lesssim l_{vdW}$



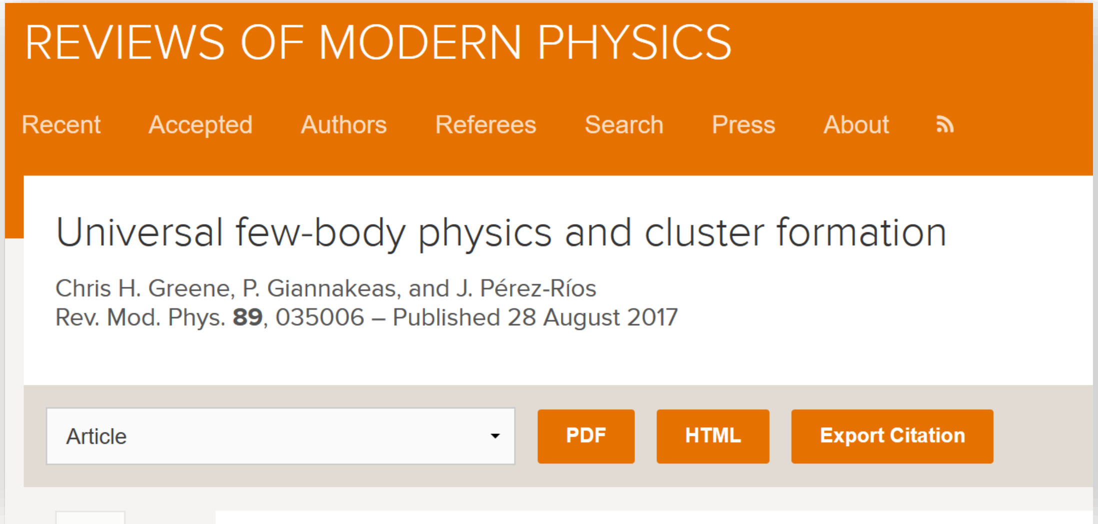
The three-body parameter



Conclusion

Universal few-body physics has been a developing field of quantum few-body physics, both theoretical and experimental, unveiling a whole collection of universal few-body states with remarkable properties.

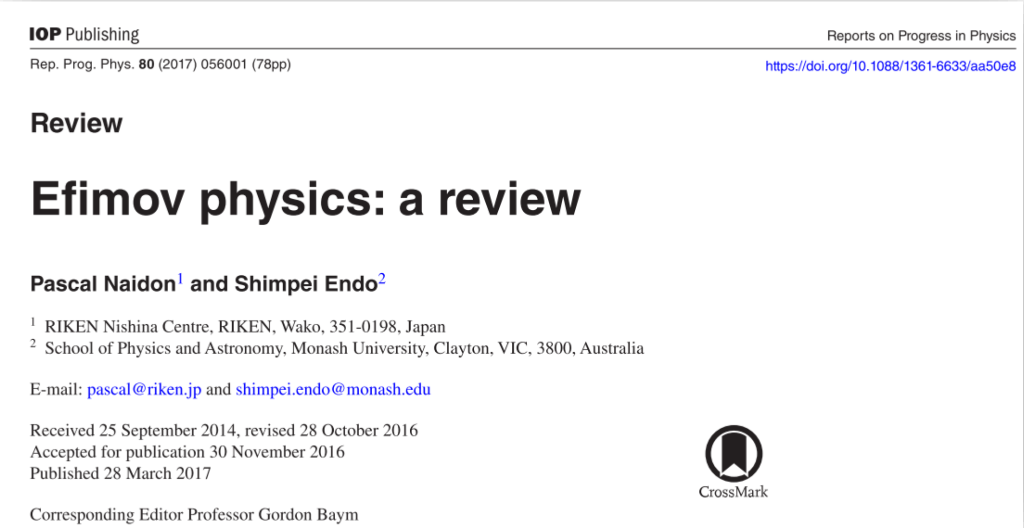
Review papers



REVIEWS OF MODERN PHYSICS

Universal few-body physics and cluster formation

Chris H. Greene, P. Giannakeas, and J. Pérez-Ríos
Rev. Mod. Phys. **89**, 035006 – Published 28 August 2017



IOP Publishing
Reports on Progress in Physics
Rep. Prog. Phys. **80** (2017) 056001 (78pp)
https://doi.org/10.1088/1361-6633/aa50e8

Review

Efimov physics: a review

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